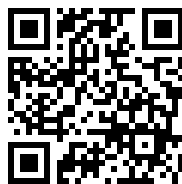


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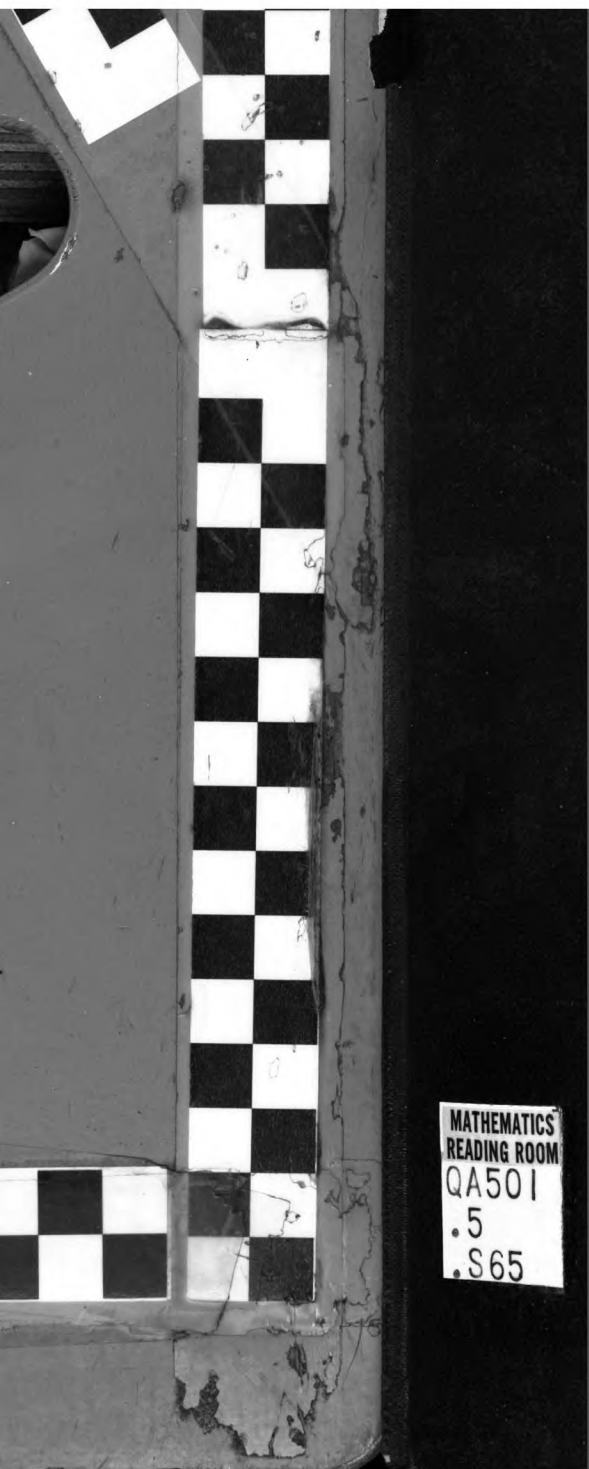
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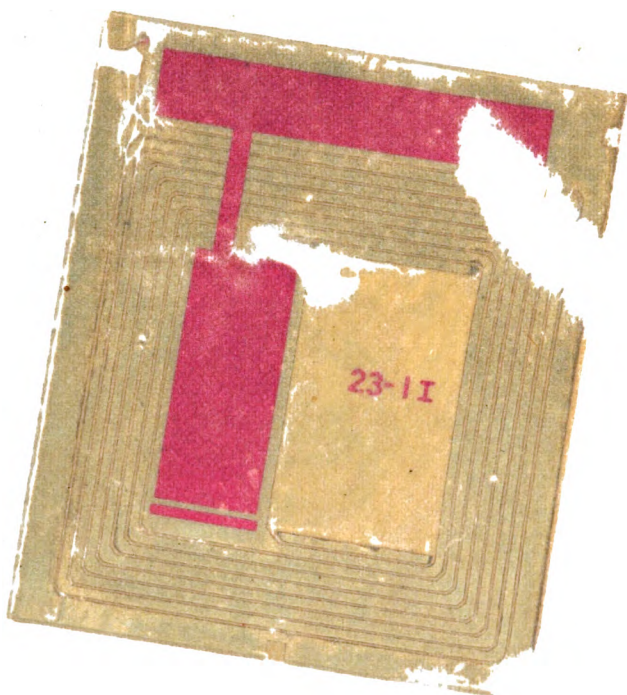
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**ELEMENTS**  
**OF**  
**DESCRIPTIVE GEOMETRY.**

WITH ITS APPLICATIONS  
TO  
SHADES, SHADOWS. AND PERSPECTIVE,  
AND TO  
TOPOGRAPHY.

BY  
FRANCIS H. SMITH, A.M.  
SUPERINTENDENT AND PROFESSOR OF MATHEMATICS OF THE VIRGINIA  
MILITARY INSTITUTE.

BALTIMORE:  
KELLY, PIET & CO.,  
174 Baltimore Street.



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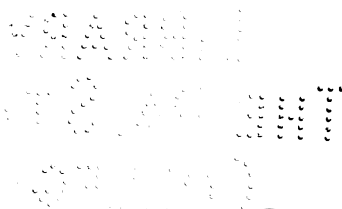
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TO  
COLONEL JOHN T. L. PRESTON,

*Professor of Latin Language and English Literature, Virginia  
Military Institute.*

---

I AM sure my associate Professors will vindicate the grounds upon which you are singled out, as one to whom I may appropriately dedicate this work.

As the originator of the scheme, by which the public guard of a State Arsenal was converted into a Military School, you have the proud distinction of being the "*Father of the Virginia Military Institute.*" You were a member of the first Board of Visitors, which gave form to the organization of the Institution; you were my only colleague during the two first and trying years of its being; and you have, for a period of twenty-eight years, given your labors and your influence, in no stinted measure, not only in directing the special department of instruction assigned to you, but in promoting those general plans of development, which have given marked character and widespread reputation to the school.

Nor is it without reason that *this* work, more than any other of my mathematical series, is dedicated to *you*. *Descriptive Geometry* was scarcely known in the schools and colleges of Virginia, when the Virginia Military Institute, by its distinctive *scientific* character, made instruction in a full course of Descriptive Geometry and its applications, a necessary part of

its programme of studies. It was thus that it was proposed, in part, to qualify the young men of Virginia for honorable *industrial* pursuits; that they might, as civil engineers, architects, machinists, and manufacturers, lend their aid in developing the wealth and industry of this Old Dominion.

No one knows the weight of all these influences, so happily exerted by you, as well as myself; and in recognizing a claim founded upon long and faithful public service, I but give expression to those sentiments of affectionate regard and esteem which I have ever cherished towards you, and which the lapse of time has served only to strengthen and confirm.

FRANCIS H. SMITH.

VIRGINIA MILITARY INSTITUTE,

November 11, 1867.

*28th Anniversary of the Va. Mil. Inst.*

## PREFACE.

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DESCRIPTIVE GEOMETRY is comparatively a modern science. The necessity of having drawing plans for workmen employed in public constructions had long since impressed architects, engineers, and master builders, with the importance of presenting, in a practical form, the established principles of Geometry, in a way to be made available in graphic representations of bodies. But it was not until towards the close of the last century that these principles were generalized and reduced to the order of a science. The honor of this important labor of genius belongs to *M. Gaspard Monge*, at that time Professor in *L'école Normale* of France, and subsequently Professor in the celebrated *École Polytechnique* of Paris. The work in which the principles of the new science were presented, was published in the proceedings of the Normal School, under the name of *Leçons de Geometrie Descriptive*, and is a model of perspicuity. In his expressive words, he says: "*C'est une langue nécessaire à l'homme de génie qui conçoit un projet, à ceux qui doivent en diriger l'exécution, et aux artistes qui doivent eux-mêmes en exécuter les différentes parties.*"

The lessons of *M. Monge* were republished by *M. Hachette*, with a supplement, in 1812. In 1818, *M. L. L. Vallée*, an élève of the Polytechnic School, extended the labors of his distinguished Professors, *Monge* and *Hachette*, and published his valuable work on Descriptive Geometry, under the authority of the Royal Academy of Sciences of the Institute of France, with the approval of the savans *Delambre*, *Prouy*, *Fourier*, and *Arago*. The principles laid down in the lessons of *Monge*, as extended by *Hachette* and *Vallée*, constitute the basis of the wonderful developments of Descriptive Geometry which have been made within the last thirty years, under the labors of *Leroy*, *Babinet*, *Lefebure de Fourcy*, *Olivier*, *Dela Gournerie*, and others; so that

the science, as it is now presented, with its various applications to Shades, Shadows, and Perspective, Stone Cutting, and Topography, constitutes an important and essential branch of study, for all who are called upon to apply graphic constructions, whether in engineering, fortification, designing, architecture, or machines.

The distinctive character of the institution to which the labors of the author are especially devoted, has always given prominence to *Descriptive Geometry* in its course of studies; and the present work, the result of thirty years experience in teaching the subject, has been prepared for the use of the cadets. It is no less designed for the general student, and it is hoped will prove an auxiliary to the study of this branch of mathematical science.

Without following the arrangement of any other work, material assistance has been drawn from all the works above cited, and especially from the valuable treatise of *Lefebure de Fourcy*.

Part I. relates to the Point, Right Line, and Plane.

Part II. will contain Curved Surfaces.

Part III. The Method of Projection on One Plane.

Part IV. Spherical Projections.

Part V. Shades, Shadows, and Perspective.

The author would acknowledge his indebtedness to *Capt. H. H. Dinwiddie*, Asst. Professor of the Va. Military Institute, for the drawings from which the plates used in this work were made.

VIRGINIA MILITARY INSTITUTE,  
November 23, 1867.

The Mathematical Series of the Virginia Military Institute, as published by Kelly & Piet, Baltimore, embrace,

*Smith's Primary Arithmetic.*

*Smith's Arithmetic for Schools and Academies.*

*Smith's Algebra.*

*Smith's Legendre's Geometry.*

*Smith's Lefebure de Fourcy's Trigonometry.*

*Smith's Biot's Analytical Geometry.*

*Smith's Descriptive Geometry.*

PART I.

THE POINT—THE RIGHT LINE—THE PLANE.



# DESCRIPTIVE GEOMETRY.

---

## INTRODUCTION.

**Definitions and Propositions of Geometry with respect to a Right Line and Plane considered in Space.**

### DEFINITIONS.

**Definition 1.** A *plane* is a surface, in which, if any two points be taken at will, the right line joining them will lie wholly in the surface.

**DEF. 2.** If a right line have only one point in common with a plane, part of the line will lie on one side, and part on the other side of the plane.

**DEF. 3.** Planes, although limited in the construction of all geometrical figures, are considered as indefinite in extent.

**DEF. 4.** A right line is said to be perpendicular to a plane, when it is perpendicular to all right lines drawn through its foot in that plane.

**DEF. 5.** If a right line be perpendicular to a plane, we say, reciprocally, that the plane is perpendicular to the line.

**DEF. 6.** Oblique lines are those which meet a plane, and are not perpendicular to it.

**DEF. 7.** A right line and plane are said to be parallel to each other, when, if either, or both, be indefinitely extended, they never meet.

**DEF. 8.** Two planes are said to be parallel, when, although produced indefinitely, they never meet.

**DEF. 9.** One plane is *perpendicular to another plane*, when it

contains all the perpendiculars to this plane, which are drawn to it through their common intersection.

DEF. 10. *The angle between two planes*, called, simply, *the diedral angle*, is the figure which is formed at the common intersection of two planes, which meet or intersect each other. The angle formed by two right lines, drawn, one in each plane, and perpendicular to the common intersection of the planes at the same point, is *the plane angle* corresponding to the *diedral angle*, and is *the measure of it*.

DEF. 11. *A solid angle*, or a *polyedral angle*, is the figure formed by several planes meeting or intersecting each other in a common point.

DEF. 12. *A triedral angle* is a solid angle formed by three plane angles; a *tetraedral angle*, one formed by four plane angles; a *pentaedral angle*, one formed by five plane angles.

### PROPOSITIONS.

NOTE. — Reference is made, for the proof of the following Propositions, where the demonstrations are not given, to "*Smith's Legendre's Geometry*."

**Proposition I.** A right line cannot be partly in a plane and partly out of it. For, by the definition of a plane, when a right line has two points in common with the plane, it lies wholly in the plane.

PROP. II. *A plane is determined in position*, when either of the following conditions is satisfied:

1st. When a right line, and a point not on the line, are known.

2d. When three sides of a triangle, or three points not in same right line, are known. (Book V. Cor. I. Prop. I., Smith's Legendre.)

3d. When two right lines which intersect each other are known. (Book V. Prop. I., Smith's Leg.)

4th. When two right lines which are parallel to each other are known. (Book V. Prop. I. Cor. II.)

PROP. III. The intersection of two planes is a right line. (Smith's Leg., Book V. Prop. II.)

PROP. IV. If a right line be perpendicular to two right lines

at their point of intersection, it will be perpendicular to the planes of these lines. (Smith's Leg., Book V. Prop. III.)

PROP. V. The perpendicular drawn from a point to a plane is the shortest line that can be drawn between the point and the plane, and measures the distance between the point and plane. (Smith's Leg., Book V. Prop. VI.)

PROP. VI. If a perpendicular be drawn from a given point to a plane, and if from the foot of the perpendicular a perpendicular be drawn to any line of the plane, and the point in which this last perpendicular meets the line be joined with the given point, this line will be perpendicular to the line in the plane. (Smith's Leg., Book V. Prop. VII.)

PROP. VII. Through a given point in space only one right line can be drawn parallel to a given line. (Smith's Leg., Book V. Prop. IX.)

It follows from this proposition, that if two right lines are parallel, the plane drawn through one of them and a point of the other will contain the whole of the other line.

PROP. VIII. Two parallel right lines are perpendicular to the same plane. (Smith's Leg., Book V. Prop. VIII.)

PROP. IX. If two right lines be perpendicular to the same plane, they are parallel to each other. (Smith's Leg., Book V. Prop. VIII. Cor. I.)

PROP. X. If two right lines be parallel to a third line, they are parallel to each other. (Smith's Leg., Book I. Prop. XXIV.)

PROP. XI. If two planes be drawn through two parallel right lines so as to intersect each other, the intersection will be parallel to these lines.

This Prop. is a corollary to Prop. X.

PROP. XII. Every right line which is parallel to another right line situated in a given plane, is parallel to the plane. (Elem. Geo., Book V. Prop. X.)

PROP. XIII. If a right line be perpendicular to a plane, every right line, perpendicular to it, will be parallel to the plane. (El. Geo., Book V. Prop. VIII.)

PROP. XIV. If a right line be parallel to a plane, and another plane be drawn through this line intersecting the given plane, the intersection of the two planes will be parallel to the given line. (El. Geo., Book V. Prop. X. Cor. I.)

PROP. XV. If a right line be parallel to a plane, any right line, drawn through any point of this plane, and parallel to the given line, will lie wholly in this plane. (El. Geo., Book V. Prop. X. Cor. II.)

PROP. XVI. If a right line be parallel to two planes which intersect each other, it will be parallel to their intersection.

For, if through a point of the intersection we draw a right line parallel to the given line, it will lie in each of the two planes, by the preceding proposition, and is therefore the intersection of these planes.

PROP. XVII. The parallels comprised between a right line and plane which are parallel, are equal.

For the plane of the parallels will intersect the given plane in a right line, which will be parallel to the given line. A parallelogram is thus formed, the opposite sides of which are equal.

COROLLARY. If a right line be parallel to a plane, every point of this line will be equally distant from the plane.

PROP. XVIII. If two planes be perpendicular to the same right line, they are parallel to each other. (El. Geo., Book V. Prop. XI.)

PROP. XIX. The intersections of two parallel planes with a third plane are parallel to each other. (El. Geo., Book V. Prop. XII.)

PROP. XX. If two planes be parallel to each other, a perpendicular to one will be perpendicular to the other.

A corollary to Prop. XVIII.

PROP. XXI. If two planes be parallel to a third plane, they are parallel to each other. (El. Geo., Book V. Prop. XV.)

PROP. XXII. Parallel lines, comprised between two parallel planes, are equal. (El. Geo., Book V. Prop. XVI.)

PROP. XXIII. If two angles, not in the same plane, have their sides parallel and lying in the same direction, they will be equal, and the planes of the angles will be parallel. (El. Geo., Book V. Prop. XVII.)

PROP. XXIV. If three parallel planes be intersected by any two right lines, in space, the lines will be cut proportionally by the planes. (El. Geo., Book V. Prop. XVIII.)

PROP. XXV. If a right line be perpendicular to a plane, any plane, drawn through this line, will also be perpendicular to the given plane. (El. Geo., Book V. Prop. XXVI.)

PROP. XXVI. If three lines, passing through the same point, be perpendicular to each other, each of these lines is perpendicular to the plane of the other two; and the three planes formed are perpendicular to each other. (El. Geo., Book V. Prop. XXVI. Scho.)

PROP. XXVII. If two planes be perpendicular to each other, every right line, drawn in one of them, perpendicular to their common intersection, will be perpendicular to the other plane. (El. Geo., Book V. Prop. XXVII.)

COROLLARY. Through a given right line, which is not perpendicular to a given plane, only one plane can be drawn perpendicular to this plane.

For this perpendicular plane must contain, besides the given line, the perpendicular let fall on the given plane, through any point whatever of the given line; and only one plane can be drawn through two right lines.

PROP. XXVIII. If two planes, which intersect, be perpendicular to a third plane, their intersection will also be perpendicular to this plane. (El. Geo., Book V. Prop. XXVIII.)

COR. I. A plane, perpendicular to two planes which intersect each other, is perpendicular to their intersection.

COR. II. If three planes be perpendicular to each other, the intersection of two of these planes is perpendicular to the third plane; and the three intersections are perpendicular to each other.

PROP. XXIX. We can always draw a right line, that shall be perpendicular to two given right lines which are not in the same plane, and this perpendicular will be the shortest distance between the two given lines. (El. Geo., Book V. Prop. XXI.)

## CHAPTER I.

### FUNDAMENTAL PRINCIPLES.

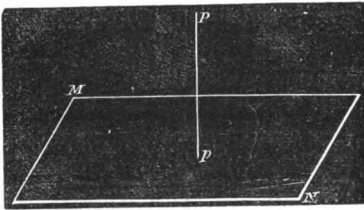
1. **Descriptive Geometry** has for its object, to explain the methods of solving problems involving three dimensions, by constructions made upon a single plane.

For this purpose, points in space are referred to two planes which intersect each other. The reference is made, by drawing, through each point, two perpendiculars; one to each plane, and then determining the points in which the perpendiculars meet these planes, after the planes have been made to coincide and form one and the same plane.

2. This process is called *the method of projections*. *Descriptive Geometry is, therefore, based upon the method of projections.*

3. *The projection of a point on a plane is the foot of the perpendicular let fall from this point on the plane.*

The plane on which the projection is made is called *the plane of projection*; and the perpendicular, which projects the point on the plane, is called *the projecting line of the point*.



Thus, if  $P$  be any point in space, and  $Pp$  the perpendicular let fall from this point on any plane  $MN$ , the point  $p$ , in which the perpendicular meets the plane, is *the projection* of the point  $P$ ;  $MN$  is *the plane of projection*; and the perpendicular

$Pp$  is *the projecting line of the point  $P$* .

4. The point  $p$  is not only the projection of the point  $P$  on the plane  $MN$ , but it is also the projection of every point of the line  $Pp$  on this plane; since the same perpendicular  $Pp$  is the projecting line of every point of the line  $Pp$ .

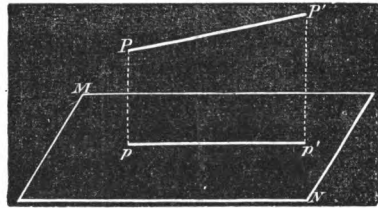
5. The point  $p$ , on the plane  $MN$ , is its own projection; for the perpendicular through  $p$  to the plane  $MN$ , evidently meets this plane at  $p$ .

6. To find the projection of a right line on a plane, we draw perpendiculars from every point of the line to the plane; the series of points, in which these perpendiculars meet the plane, will be the projection of the line. This projection is necessarily a right line; since all the perpendiculars are contained in the same plane which passes through the given line, (Int. Prop. XXVII.,) and the intersection of this plane with the given plane is a right line. (Int. Prop. III.)

Since two points determine the position of a right line, it is only necessary to construct the projections of two points of the line, to find the projection of the line.

7. The projection of a right line on a plane is, therefore, the intersection with this plane, of a plane drawn through the line, and perpendicular to the plane on which the projection is made.

Thus, if  $PP'$  be the line to be projected on the plane  $MN$ , the series of perpendiculars  $Pp$ ,  $P'p'$ , &c. will form a plane perpendicular to the plane  $MN$ ; and the series of points  $p$ ,  $p'$ , &c. will form a right line in which this plane will intersect the plane  $MN$ ; and this line is the projection of the line  $PP'$ .

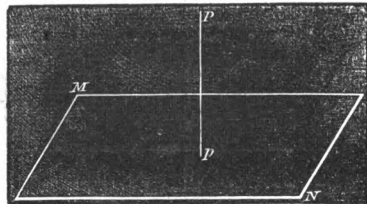


8. The plane which projects a right line on a given plane, is called the projecting plane of the line.

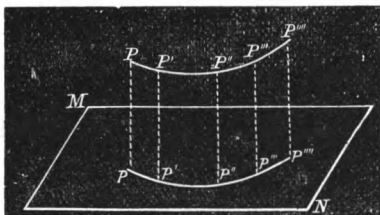
9. If the line to be projected be perpendicular to the plane of projection, all the perpendiculars, drawn through its various points, will coincide with the line itself; and the intersection with the plane of projection, which is, in this case, the projection of the line, will be a point.

Thus, if  $Pp$  be perpendicular to the plane  $MN$ , its projection on the plane  $MN$  is the point  $p$ .

10. The projection of a curve line on a plane is found, by drawing lines through every point of the curve, perpendicu-



lar to this plane, and joining the series of points in which the perpendiculars meet the plane. The line thus formed is the *projection of the curve*. These perpendiculars being parallel, (Int. Prop. IX.,) they will form a surface to which the general name of *cylinder* is given. *The cylinder which is thus used to project the given curve on a plane, is called the projecting cylinder of the curve.*

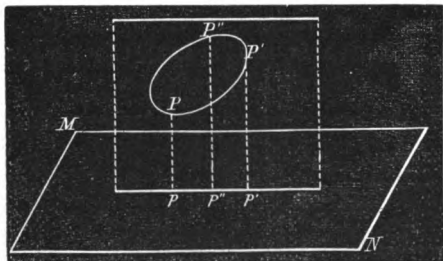


Thus, if  $P, P', P'',$  &c. be the given curve, and  $MN$  the plane of projection, the series of perpendiculars  $Pp, P'p',$  &c., will form the surface of the projecting cylinder, and their intersection with the plane  $MN$  will give  $pp'p''$

&c., as the projection of the curve  $PP'P'',$  &c.

It is evident that any curve, traced on the surface of the projecting cylinder, will be projected on the plane  $MN$ , in the same curve  $pp'p'',$  &c.

11. If the given curve be a *plane* curve, and its plane be perpendicular to the plane of projection, the projecting lines of its several points will coincide with the plane of the curve, and the projecting cylinder will be thus reduced to a plane. The intersection of this plane with the plane of projection will be the projection of the curve. Since this intersection is a right line (Int. Prop. III.), we see, *that the projection of a plane curve which is perpendicular to the plane of projection is a right line.*



Thus, if  $PP'P'',$  &c. be a plane curve, whose plane is perpendicular to  $MN$ , its projection will be the right line  $pp'p'',$  &c.

12. If the plane of the curve be parallel to the plane of projection, its projection will be a

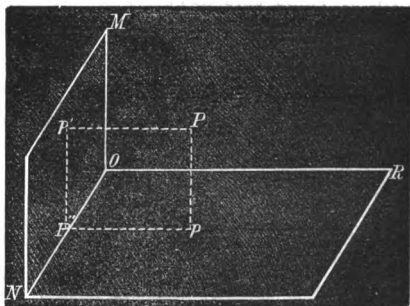
curve identical with itself.

13. These principles being established, it follows, *that a point*

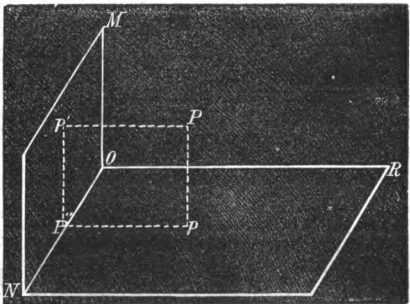
*in space will be completely determined, by the method of projections, when we know its projections on two given planes which intersect each other.*

For, the point must be found, at the same time, on the two projecting lines of the point; that is, on two lines drawn through the two assumed projections of the point perpendicular, respectively, to the two planes; and these lines can only intersect each other in one point.

Thus, if  $p'$  and  $p$  be the two given projections of the point  $P$ , on the planes  $MN$  and  $NR$ ; the point  $P$  must be on the projecting line  $Pp'$ , and also on the projecting line  $Pp$ ; and since  $Pp'$  and  $Pp$  are the projecting lines of the same point, they must intersect each other, and their intersection fixes the position of the point  $P$ .



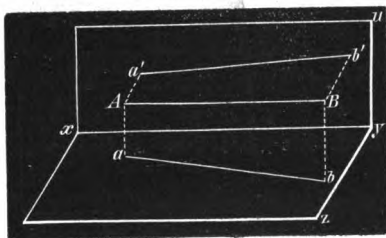
14. It does not follow from this that any two points assumed, at will, on two planes, will be the projections of the same point. For, if  $p'$  and  $p$  be the two projections of the point  $P$ , on two given planes which intersect each other in the line  $NO$ ; then, if a plane  $Ppp''p'$  be drawn through the projecting lines  $Pp$  and  $Pp'$ , intersecting the plane  $MN$  in the line  $p'p''$ , and the plane  $NR$  in the line  $pp''$ , and the intersection  $NO$  of the two given planes in the point  $p''$ ; since the plane  $Ppp''p'$  contains the perpendicular  $Pp'$  to the plane  $MN$ , it is perpendicular to the plane (Int. Prop. XXV.); for a like reason, it is perpendicular to the plane  $NR$ . It is therefore perpendicular to the common intersection  $NO$ , (Int. Prop. XXVII. Cor. 1.) But if



$NO$  be perpendicular to the plane  $Ppp''p'$ , it is also perpendicular to the lines  $pp''$  and  $p'p''$ , drawn in this plane, through the point  $p''$ . (Int. def. 4.) Hence, we have the condition that two points, situated, respectively, in two planes which intersect each other, may be the projections of the same point in space; viz: that the perpendiculars, drawn through these points to the common intersection of the two planes, shall meet this intersection in the same point.

15. A right line in space is determined, when we know its projections on two planes which intersect each other.

For, if through each projection a plane be drawn, perpendicular

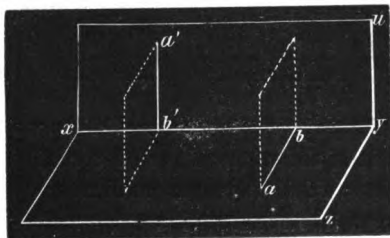


to the plane of this projection; these planes will, each, contain the line in space, and their intersection must necessarily determine it. Thus, let  $ab$  and  $a'b'$  be the projections of a right line  $AB$ , on the planes  $xyz$  and  $xyu$ ; now, if a plane be

drawn through  $ab$  perpendicular to the plane  $xyz$ , and another plane be drawn through  $a'b'$  perpendicular to  $xyu$ , the intersection  $AB$  of these two planes will determine the right line in space, of which  $ab$  and  $a'b'$  are the projections.

16. When the assumed projections of the line are perpendicular to the common intersection of the two planes, and meet that intersection in different points, they cannot be the projections of the same right line.

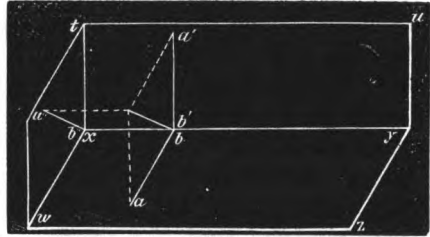
For, if  $ab$  be the assumed projection on the plane  $xyz$ ; and



$a'b'$  that on the plane  $xyu$ ; then, the plane through  $ab$ , perpendicular to the plane  $xyz$ , will be perpendicular to the line  $xy$ , (Int. Prop. XXVIII. Cor. 1,) and the plane through  $a'b'$  will also be perpendicular to  $xy$ . Hence, these planes, being

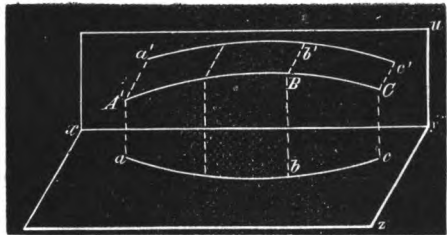
perpendicular to the same right line, will be parallel to each other (Int. Prop. XVIII.), and cannot therefore intersect.

17. If the two assumed projections of the line, still perpendicular to  $xy$ , meet the common intersection of the two planes of projections in the same point, these two projections will not fix the position of the line in space; and the line will be undetermined. For, in this case, the plane of the two projections  $ab$  and  $a'b'$  will be perpendicular to the two planes  $xyz$  and  $xyu$ , at the same point; and there will be but one projecting plane. Any line, therefore, situated in the plane  $a'b'a'$  will have  $ab$  and  $a'b'$  for its projections. The line in space will not be determined, unless we know its projections  $a''b''$  on a third plane  $txw$ , which does not pass through the line  $xy$ .



18. In general, any line in space, whether curved or right, is determined, when we know its projections on two planes which intersect each other.

For, if  $abc$  and  $a'b'c'$  be the projections of the curve  $ABC$  on the planes  $xyz$  and  $xyu$ ; then, if lines be drawn through the various points of each projection, perpendicular to the plane on which the projection is made, they will form the surfaces of two cylinders, perpendicular, respectively, to these planes; and the intersection of these cylinders will determine the curve in space.



19. If the cylinders do not intersect, the assumed projections will not belong to the same curve.

20. If the given curve be a plane curve, whose plane is perpendicular to the common intersection of the two planes of projection; the two projecting cylinders will reduce to a plane, which will be the same with that of the curve. The curve is then undetermined. The indetermination in this case is similar to

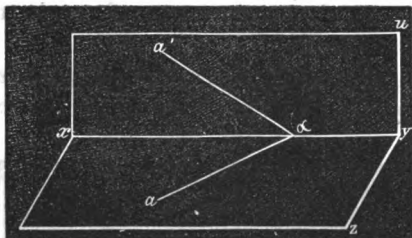
that which has just been noticed with reference to a right line (17).

21. *A plane is determined when we know its intersections with any two planes which intersect each other.*

This is evident, since a plane is determined when it passes through two right lines which intersect each other. (Int. Prop. II. 3.)

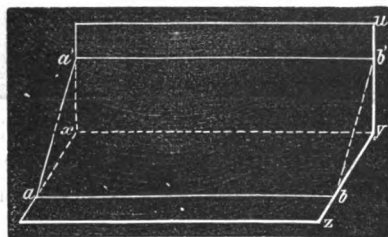
22. Since any three points in space, which are not in the same right line, also determine a plane (Int. Prop. II. 2); we might fix the position of a plane by the method of projections, by employing the projections of three points of the plane, or of three right lines which connect these points.

23. *The intersection of a plane with either plane of projection is called its trace on that plane.*



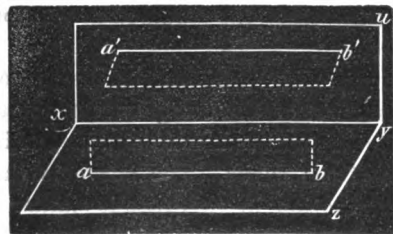
24. If a plane intersect the common intersection of the two planes of projection, the point of intersection is evidently common to the two traces. Thus, the point  $a$ , in which the plane  $a a'$  meets the intersection  $xy$ , is a point

of the trace  $a a$ , and also of the trace  $a' a$ .



25. If a plane be parallel to the common intersection of the two planes of projection, its two traces will also be parallel to it.

Thus, the traces  $ab$  and  $a'b'$  of the plane parallel to  $xy$ , are parallel to  $xy$ .



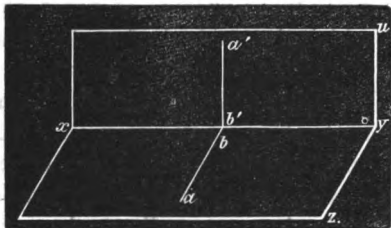
26. If a plane be parallel to one of the planes of projection, its trace on the other plane will be parallel to the common intersection of the two planes of projection; and this trace will

be sufficient to determine the position of the plane. (Int. Prop. XIX.)

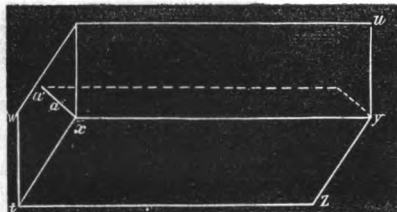
Thus,  $a'b'$ , parallel to  $xy$ , is the trace of a plane parallel to the plane  $xyz$ , and fixes the position of this plane; while  $ab$ , parallel to  $xy$ , is the trace of a plane parallel to  $xyu$ .

27. If a plane be perpendicular to the common intersection of the two planes of projection, its traces will also be perpendicular to their intersection.

Thus,  $ab$ , in the plane  $xyz$ , and  $a'b'$ , in the plane  $xyu$ , are both perpendicular to  $xy$ ; and are the traces of a plane perpendicular to  $xy$ .



28. Finally, if a plane pass through the common intersection of the two planes of projection, its traces will coincide with this intersection, and the plane will be undetermined; unless we have its trace on a third plane which does not pass through this intersection.



Thus, the trace  $a'a$ , on a plane  $xtw$ , will fix the position of a plane passing through  $xy$ .

29. The above principles are independent of the inclination of the planes of projection. It is found most convenient in practice to take the planes of projection at right angles to each other; one of them being horizontal, and the other vertical.

30. The horizontal plane thus assumed as one of the planes of projection, is called *the horizontal plane of projection*; and the vertical plane is called *the vertical plane of projection*.

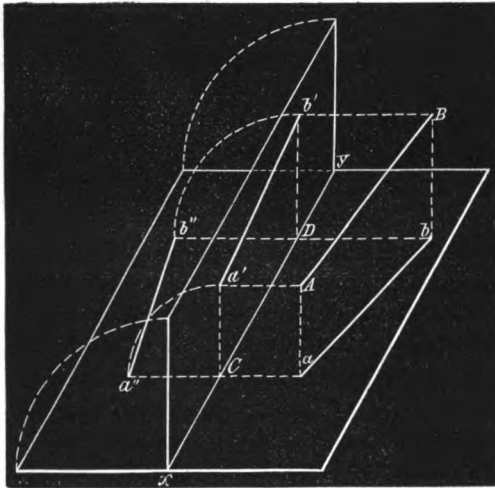
31. The common intersection of the vertical and horizontal planes of projection is called *the ground line*.

32. The projections of all points and lines, and the traces of all planes, made on these planes of projection, take the name of the plane on which the projection or trace is made. Thus, if the projection or trace be on the horizontal plane of projec-

tion, it is called the *horizontal projection* of the point or line; or the *horizontal trace* of the plane; and, reciprocally, the *vertical projection*, or the *vertical trace*, if the representation be made on the vertical plane of projection.

33. In order that the two projections which determine the positions of points, lines, &c., in space, may be represented on a *single plane* or *sheet of paper*, as is required in all drawings in which the method of projections is used; the vertical plane of projection is revolved about the ground line, as an axis, until it coincides with the horizontal plane of projection, so as to form with it a *single plane*. By this means, the two projections which, in imagination, are conceived to be represented on *two* planes, are in reality constructed on *one*; that is, the vertical projections are really made on the horizontal plane.

Thus, the vertical projection  $a' b'$ , of the right line  $AB$ , is not executed on a plane that is really vertical; but, the vertical



plane of projection being supposed to revolve about the ground line  $xy$ , until it coincides with the horizontal plane; it is in this position of the vertical plane that we make the construction of the projection  $a' b'$ .

34. Independently of the convenience of the construction

which this arrangement produces, it very materially abridges the labor of finding the projections. For, if  $a$  and  $a'$  be the two projections of the point  $A$ , the plane drawn through  $Aa$  and  $Aa'$ , will be perpendicular at the same time to the two planes of projection, since it contains a line perpendicular to each (Int. Prop. XXV.); it will therefore be perpendicular to the common intersection  $xy$  (Int. Prop. XXVIII.); and the lines  $aC$  and  $a'C$  in which it intersects the planes of projection will also be perpendicular to  $xy$ . But, when the vertical plane is revolved about  $xy$ , as an axis, the line  $a'C$  does not cease being perpendicular to  $xy$ , during the revolution; and is perpendicular to  $xy$ , when it takes the position  $a''C$ . Then, the lines  $aC$  and  $a''C$ , being both perpendicular to the same line  $xy$ , and passing through the same point  $C$ , are the prolongation, one of the other. The same reasoning applies to the lines  $bD$  and  $b''D$ , with respect to the point  $B$ . From which it follows, that if the horizontal projection of a point be known, the vertical projection will be found, after the vertical plane has been revolved to coincide with the horizontal plane, on a right line drawn through the horizontal projection of the point perpendicular to the ground line; and, conversely, if the vertical projection of a point be known, the horizontal projection will be on a right line drawn through the vertical projection of the point perpendicular to the ground line.

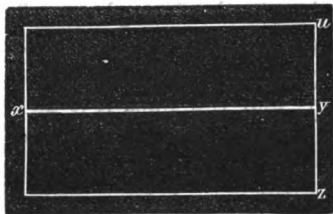
Hence, we have this important principle, *that the two projections of a point are always found in the same right line drawn perpendicular to the ground line.*

35. Since  $C=Aa'$ , and  $Db=Bb'$ ; it follows, *that the distance of the horizontal projection of a point from the ground line, is always equal to the distance of the point in space from the vertical plane of projection.*

36. Since  $a''C=a'C=Aa$ , and  $b''D=b'D=Bb$ ; it follows, *that the distance of the vertical projection of a point from the ground line is always equal to the height of the point in space above the horizontal plane of projection.*

37. If the observer be in front of the vertical plane, and above the horizontal, it is found most convenient to cause the revolution of the vertical plane to be made *from* the observer, and not *towards* him; since, under this supposition, the projection will,

in general, be made on opposite sides of the ground line; thus avoiding confusion in the drawing.



If  $xy$ , therefore, be the ground line; the vertical plane, after the revolution, will lie above the line  $xy$ , and will be indicated by the plane  $xyu$ ; and it is in this position of the planes of projection that the constructions will be made.

38. The condition that the planes of projection are perpendicular to each other, leads to the following important principles.

1°. If a point or line be situated in one of the planes of projection, the projection on the other plane will be in the ground line.

For, the projecting perpendiculars are all contained in the plane in which the point or line is situated; and must, therefore, meet the other projection in the ground line.

2°. If a right line be parallel to the horizontal plane of projection, its vertical projection will be parallel to the ground line.

For, the projecting perpendiculars, which determine the vertical projection, will be contained in a plane parallel to the horizontal plane; and the intersection of this plane with the vertical plane of projection, which determines the vertical projection of the line, must be parallel to the ground line. (Int. Prop. XIX.)

3°. If a right line be parallel to the vertical plane of projection, its horizontal projection will be parallel to the ground line.

For, the horizontal projecting lines will be contained in a plane parallel to the vertical plane of projection, and the intersection of this plane with the horizontal plane of projection, which determines the horizontal projection, will be parallel to the ground line.

4°. If a right line be parallel to the two planes of projection, it will also be parallel to the ground line, and its projections on both planes will be parallel to the ground line.

This follows from the two preceding principles.

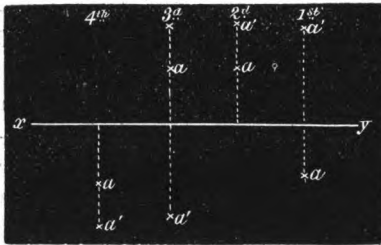
5°. If a plane be perpendicular to one of the planes of projection, its trace on the other will be perpendicular to the ground line; that is, a plane perpendicular to the horizontal plane of projection, will have

*its vertical trace perpendicular to the ground line; and a plane perpendicular to the vertical plane of projection will have its horizontal trace perpendicular to the ground line.*

39. If the planes of projection be considered as indefinite in extent, as is usually the case with all lines and planes in Descriptive Geometry, there will be formed *four diedral angles*, in either of which a point or line may be situated, when its projections are determined. The *first* angle is usually regarded as that angle which lies in front of the vertical plane and above the horizontal; the *second* is behind the vertical and above the horizontal; the *third* is behind the vertical and below the horizontal; and the *fourth* is in front of the vertical and below the horizontal.

40. The principles which have been established for fixing the position of the projections of a point in one angle, are equally applicable to all, and may be readily extended to them. All points, situated in the 1st and 2d angles, will be projected on the vertical plane on that part of the vertical plane which lies *above* the horizontal plane, and their projections will necessarily fall above the ground line when the vertical plane is revolved to coincide with the horizontal plane; while points situated in the 3d and 4th angles, will be vertically projected on that part of the vertical plane which lies below the horizontal plane, and these projections will be found below the ground line, since that part of the vertical plane which lies below the horizontal plane, will fall below the ground line, when the revolution of this plane to coincide with the horizontal plane has been effected. The *horizontal* projections of points, in the 2d and 3d angles, will be made on that part of the horizontal plane which is *behind* the vertical plane; and the horizontal projections of points, situated in the 1st and 4th angles, on that part of the horizontal plane which is *in front* of the vertical plane. As the horizontal plane does not change its position during the revolution of the vertical plane, the horizontal projections of all points in the 2d and 3d angles will be *above* the ground line; and those of points in the 1st and 4th angles, *below* the ground line.

41. The figure will explain the positions of the projections in each angle.



**1st Angle.** The vertical projection  $a'$ , above, and the horizontal projection  $a$ , below, the ground line.

**2d Angle.** The two projections  $a$  and  $a'$  above the ground line.

**3d Angle.** The horizontal projection  $a$ , above the

ground line, and the vertical projection  $a'$  below.

**4th Angle.** Both projections  $a$  and  $a'$  below the ground line.

42. The positions of the projections of lines and curves will be found, in like manner, in whichever of the diedral angles the lines or curves which they represent may be situated.

43. Since the position of the planes of projection may be assumed, at will, and is generally taken so as to give the simplest solution to the problem, the points, lines, &c., are usually considered in the 1st angle; and it is under this supposition that the constructions will be generally made in this treatise. At the same time, it will supply an admirable practice for the student, to make the constructions, for the various problems, in either of the diedral angles.

44. Descriptive Geometry, by the principles thus explained, becomes a species of *sign language*, so that by means of the *drawings* which are made, we may have a tolerably accurate idea of the process used in the construction of the problems.

45. In making the various drawings used in Descriptive Geometry, and in the references made to them in the text, certain conventional signs are used. Thus:

*Large letters, A, B, C, &c.,* represent points in space, and are seldom used in the figures.

*Small letters, a, b, c, &c.,* without accents, represent horizontal projections; while the same letters, with accents,  $a'$ ,  $b'$ ,  $c'$ , &c., denote corresponding vertical projections.

The *Greek letters,  $\alpha$ ,  $\beta$ ,  $\gamma$ , &c.,* denote points on the ground line.

The ground line is always denoted by  $xy$ .

The *point*  $(a, a')$  means the point in space, the projections of which are  $a$  and  $a'$ .

The *line* ( $a b, a' b'$ ) means the line in space, of which the projections are  $a b$  and  $a' b'$ .

The *plane* ( $a a a'$ ) means the plane whose traces are  $a a$  and  $a' a$ .

*Given* and *required* lines are always drawn *full*. *Auxiliary* lines used in the construction are *dotted*.

Traces of auxiliary planes and invisible traces are drawn *broken*.

Finally, when we say a *point*, *line*, or *plane* is given, we always mean, that *the projection* of the point or line, or the traces of the plane are given. As also, when it is proposed to determine a point, line, or plane, we always mean, that we wish to determine the projections of the point or line, or the traces of the plane.

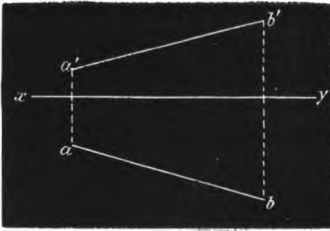
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## CHAPTER II.

### PROBLEMS—RIGHT LINE AND PLANE.

#### PROBLEM I.

46. *Two points being given by their projections, find the projections of the right line joining these points.*



Let  $(a, a')$   $(b, b')$  be the given points;  $xy$  the ground line. Since the required line has to pass through the two given points, the projections of this line must pass through the projections of these points. Therefore the line  $ab$ , joining the horizontal projections  $a$  and  $b$ , is the horizontal projection of the required line; and the line  $a'b'$ , joining the two vertical projections  $a'$  and  $b'$ , is the vertical projection of this line.

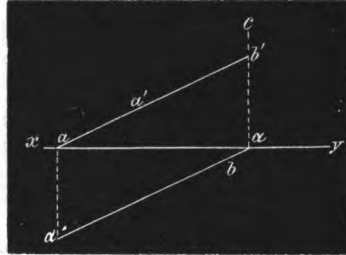
**Remark.**—By changing the position of either or both of the given points, with respect to the four diedral angles of the planes of projection, variations may be given to the problem. The student can make these variations.

#### PROBLEM II.

47. *A right line being given by its projections, find the traces of its two projecting planes.*

Let  $(ab, a'b')$  be the given line, and  $xy$  the ground line. In general, each projecting plane will have two traces, which will usually intersect each other in the ground line. The horizontal trace of the horizontal projecting plane coincides with the

horizontal projection of the line itself (7), and its vertical trace is perpendicular to the ground line (38-5). The horizontal trace of this projecting plane is, therefore,  $ab$ ; and  $a'c'$  perpendicular to the ground line at  $a$  is the vertical trace. In like manner, the vertical trace of the vertical projecting plane coincides with the vertical projection of the line, while its horizontal trace is perpendicular to the ground line. The vertical trace of this plane is, therefore,  $a'b'$ , and its horizontal trace is  $a'c$ , perpendicular to the ground line at  $a'$ .

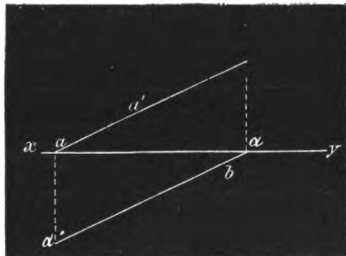


### PROBLEM III.

48. *A right line being given by its projections, find the points in which it meets the planes of projection.*

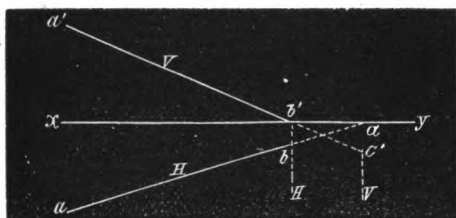
The given line in space is the intersection of its two projecting planes (15). It is, therefore, common to both of these planes, and must meet the vertical plane of projection at the intersection of the vertical traces of its two projecting planes; and it must meet the horizontal plane of projection at the intersection of the horizontal traces of the two projecting planes of the line.

Let  $(ab, a'b')$  be the given line;  $xy$  the ground line.  $ab$  and  $a'a'$  are the horizontal traces of the two projecting planes (47), and  $a'b'$  and  $a'b'$  are the vertical traces of these planes. Hence,  $a$  is the point in which the given line meets the horizontal plane of projection, and  $b'$  is the point in which it meets the vertical plane of projection.

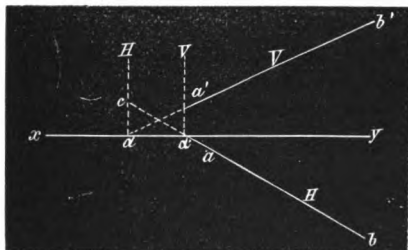


49. *Variation 1°.* Let  $(ab, a'b')$  be the given line. The horizontal traces of the two projecting planes are indicated in

the figure by the letter  $H$ ; and the vertical traces by the letter  $V$ .

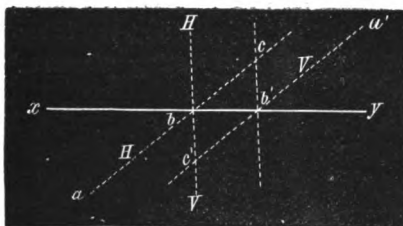


The given line meets the horizontal plane at  $b$ , at the distance  $b b'$  in front of the vertical plane (35); and it meets the vertical plane at  $c'$ , at a distance  $a c'$  below the horizontal plane (36).



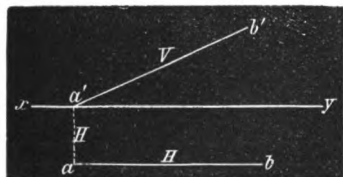
50. **Var. 2°.** Let  $(a b, a' b')$  be the given line. The horizontal traces of the two projecting planes are marked  $H$ , and the vertical traces are marked  $V$ . The line meets the horizontal plane at  $c$ , at the distance  $a c$  behind the vertical plane;

and it meets the vertical plane at  $a'$ , at the distance  $a' a'$  above the horizontal plane.



51. **Var. 3°.** Let  $(a b, a' b')$  be the given line. It meets the horizontal plane behind the vertical plane, at  $c$ ; and the vertical plane below the horizontal at  $c'$ .

52. **Var. 4°.** Let the horizontal projection,  $a b$ , be parallel to the ground

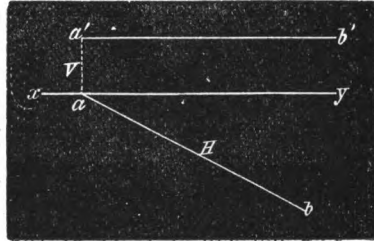


line, and  $(a b, a' b')$  the given line. The line in space is parallel to the vertical plane of projection (38-3), and will not therefore meet this plane. It meets the horizontal plane at  $a$ .

53. **Var. 5°.** If  $(a b, a' b')$  be the given line, the vertical projection being parallel to the ground line. The line in space is parallel to the horizontal plane (38-2), and does not, there-

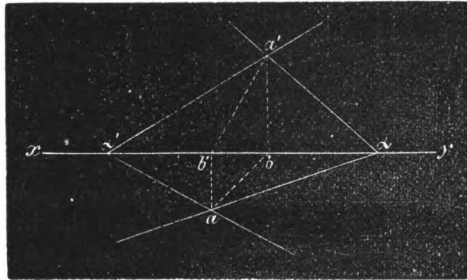
fore, meet this plane. It meets the vertical plane at  $a'$ .

54. Finally, if the line be parallel to the ground line, its projections will also be (38-4), and the problem becomes impossible, since the line does not meet either plane of projection.



#### PROBLEM IV.

55. Two planes which intersect each other being given by their traces, find the projections of their line of intersection.



Let  $(a\ a' a')$  and  $(a\ a' a')$  be the given planes;  $a$ , the intersection of their horizontal traces, and  $a'$  the intersection of their vertical traces. The line of intersection being a line of both planes, must meet the planes of projection in the traces of these planes. It meets the vertical plane at  $a'$ , and the horizontal plane at  $a$ . The points  $a$  and  $a'$  are, therefore, points on the line of intersection of these planes, and the right line which joins them is evidently the intersection of these planes. It is now required to find the projections of this line. These projections will be determined when we have found the projections of two points of the line (6). The point  $a$ , being in the horizontal plane, is its own *horizontal* projection (5), and its vertical projection is at  $b'$  where a perpendicular through  $a$  meets the ground line. In like manner, the point  $a'$ , being in the vertical plane of projection, is its own *vertical* projection; and its horizontal projection

C

is at  $b$ , where a perpendicular through  $a'$  meets the ground line. The line  $ab$ , which connects the two horizontal projections of two points of the line of intersection, is the horizontal projection of this line; and the line  $a'b'$ , connecting the two vertical projections of these points, is the vertical projection of this line. ( $ab, a'b'$ ) is, therefore, the required line of intersection.

56. *Variation 1°.* Let one of the given planes be perpendicular to one of the planes of projection; the vertical plane, for example.

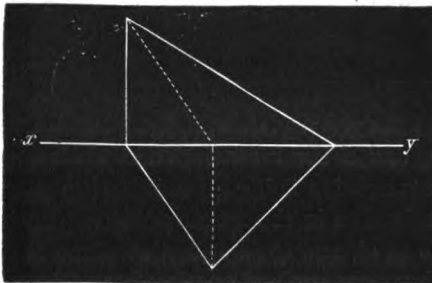
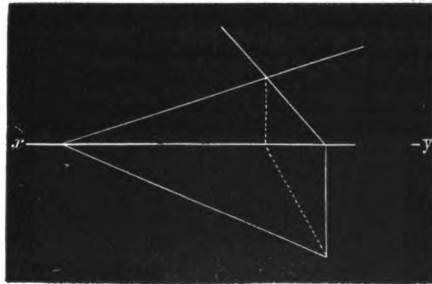
The horizontal trace of the perpendicular plane will then be perpendicular to the ground line (38-5); and the ver-

tical projection of the line of intersection will evidently coincide with the vertical trace of this plane.

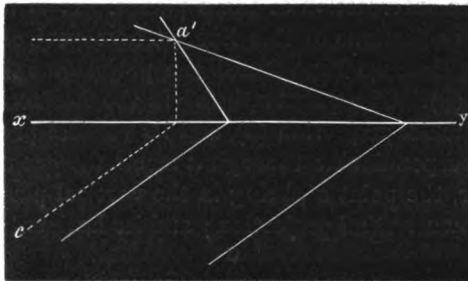
If one of the planes be perpendicular to the horizontal plane of projection, its vertical trace will be perpendicular to the ground line (38-5); and the horizontal projection of the line of intersection will evidently coincide with the horizontal trace of this plane.

The construction can readily be made.

57. *Var. 2°.* Let the horizontal traces of the



given planes be parallel to each other.

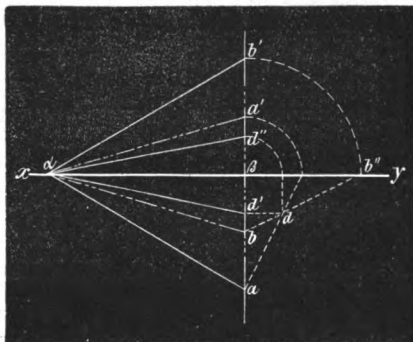


In this case, the intersection of the two planes will be parallel to these traces (Int. Prop. XI.); its horizontal projection will also be parallel to them (Int. Prop. XIV.); and its vertical projection will be parallel to the ground line (38-2.) We have  $c\alpha$ , for the horizontal, and  $c'a'$ , for the vertical projections of the line of intersection.

58. **Var. 3°.** Let the given planes intersect the ground line at the same point.

In this case, the general construction fails, and another must be applied. Let  $(aa')$  and  $(bb')$  be the two given planes. If we draw any auxiliary plane, cutting the two given planes in any two lines; these lines of intersection will intersect each other in a point that will be common to the two given planes, and will therefore be a point in their line of intersection.

If the projections of this point be joined with the point in which the planes meet the ground line, we shall have the projections of the line of intersection.



It is usual in such cases to draw the auxiliary plane perpendicular to the ground line. Let  $a\beta$  and  $a'\beta$  be the traces of such a plane. This plane intersects the plane ( $\alpha\alpha'$ ) in a line which meets the vertical plane of projection at  $a'$ , and the horizontal plane at  $a$ . It intersects the plane ( $\beta\beta'$ ) in a line which meets the vertical plane at  $b'$ , and the horizontal plane at  $b$ . Let the auxiliary plane be revolved about its horizontal trace  $a\beta$ , as an axis, until it coincides with the horizontal plane. The points  $a$  and  $b$ , being in the axis, will remain fixed during the revolution. The points  $a'$  and  $b'$ , being in the vertical plane, will describe the arcs of circles, with  $\beta a'$  and  $\beta b'$  as radii, around  $\beta$  as a centre, and will be found after the revolution in the ground line at  $a''$  and  $b''$ .  $aa''$  and  $bb''$  will be the revolved positions of the lines cut out of the two given planes by the auxiliary plane. These lines intersect each other at  $d$ . The point  $d$  is, therefore, one point, in its revolved

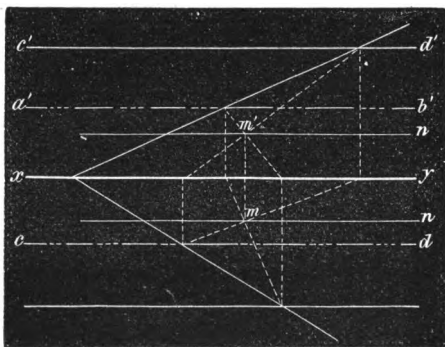
position, of the line of intersection of the two planes. In the counter-revolution of the auxiliary plane, the point  $d$  will be horizontally projected at  $d'$ , and vertically projected at  $d''$ . Hence,  $d'a$  and  $d''a$  are the projections of the line of intersection required.

59. *Var. 4°.* Let the given planes be parallel to the ground line.

In this case, the traces of the two planes will be parallel to the ground line, and the line of intersection will be also. If we

draw an auxiliary plane, as in the last article, the intersection of the two planes is readily determined.

To illustrate the general principle more fully, we have drawn the auxiliary plane, in the figure, oblique to the ground line; and we find  $mn$  and  $m'n'$  for the projections of the



line of intersection.

### PROBLEM V.

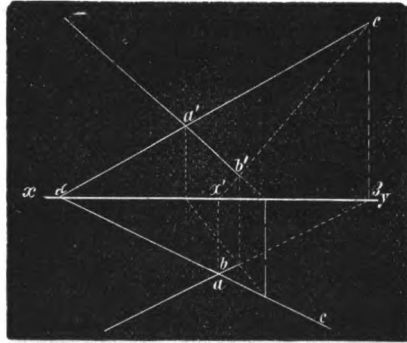
60. A plane being given by its traces, and a line not parallel to it by its projections, find the projections of the point in which the line meets the plane.

Let  $(ab, a'b')$  be the given line, and  $ca c'$  the given plane.

The problem may be solved by drawing any plane through the given line, and finding its intersection with the given plane, as in Art. 55, and then determining the point in which the given line meets the line of intersection. This point will be the point in which the given line meets the given plane. The construction will be made in the simplest manner, if the auxiliary plane be drawn perpendicular to either plane of projection. We will first make the construction under this supposition.

Let the auxiliary plane be the horizontal projecting plane of the given line. Its horizontal trace will coincide with the horizontal projection of the given line, and is  $a\beta$ ; and its vertical

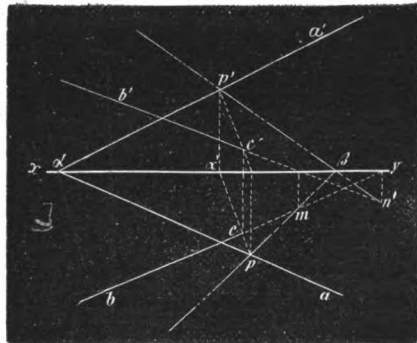
trace is  $\beta'c'$ , perpendicular to the ground line. The line of intersection of this plane and the given plane has  $a\beta$ , for its horizontal, and  $a'c'$ , for its vertical projection (55). The vertical projection of the point in which the given line meets the given plane must be found somewhere on the vertical projection  $a'c'$ . It must also be found in the vertical projection  $a'b'$  of the given line. It must therefore be found at  $b'$  where these two projections intersect. The horizontal projections of the required point must be found in the horizontal projection of the given line: that is, in  $ab$ ; it must also be in the perpendicular to the ground line through  $b'$  (34). It is therefore at  $b$ , the intersection of these lines. The required point is therefore  $(b, b')$ .



The horizontal projection  $b$  might be found, by taking, as an auxiliary plane, the vertical projecting plane of the given line. We have  $a's'$  and  $s'c'$  for its traces; and  $a'b'$  and  $bc$  will be the projections of the line of intersection of this auxiliary plane with the given plane. The point  $b$ , in which the horizontal projection of this intersection meets the horizontal projection of the given line, is evidently the horizontal projection of the point in which the given line meets the given plane.

**61. Variation 1°.** Let the construction be made under the supposition that the auxiliary plane is any plane whatever.

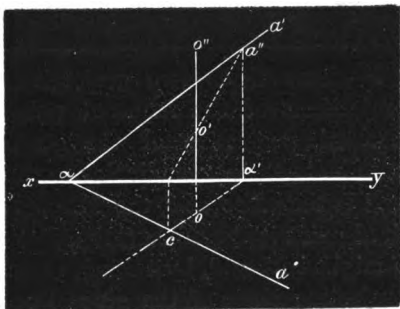
In order that the auxiliary plane shall pass through the given line, its traces must pass through the points in which this line meets the planes of projection, that is, through



$m$  and  $n'$  (48). Through  $m$  and  $n'$  draw the lines  $\beta p$  and  $\beta p'$ , to any point  $\beta$  taken on the ground line. We know that  $(p\beta p')$  is a plane passing through the given line, since it has two points in common with it (Int. Def. 1). Let us now determine the intersection of this auxiliary plane with the given plane (55). We have  $p c$  and  $p' c'$  for the projections of the line of intersection, and  $(c, c')$  as the point in which the given line meets the given plane.

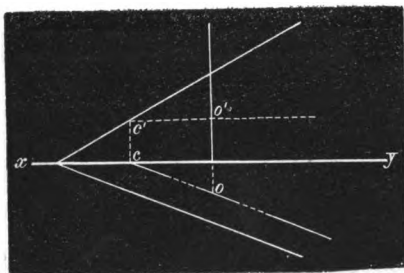
62. *Var. 2°.* Let the given line be perpendicular to one of the planes of projection.

If it be perpendicular to the horizontal plane, its horizontal



projection will be the point  $o$ , and its vertical projection will be  $o' o''$  perpendicular to the ground line (9). In this case, the auxiliary plane is perpendicular to the horizontal plane (Int. Prop. XXV.); its vertical trace must therefore be perpendicular to the ground line (38-5). Its hor-

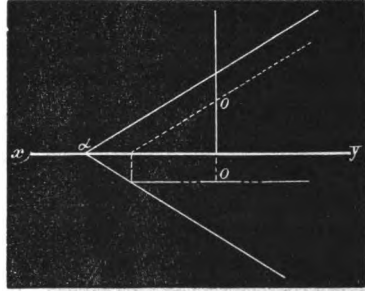
izontal trace may be any line passing through  $o$ , and  $(c a' a'')$  will be the auxiliary plane. The problem is thus reduced to finding the intersection of the plane  $(c a' a'')$  with the given plane, and determining the point  $(o, o')$  in which the line of intersection meets the given line, as in the general construction.



63. *Var. 3°.* We might modify the last construction by taking the horizontal trace of the auxiliary plane parallel to the horizontal trace of the given plane. In this case, the intersection of the two planes will be parallel to the horizontal trace (Intr.

Prop. XVI.), and its vertical projection will be parallel to the ground line (38-2); and  $(o, o')$  will be the required point.

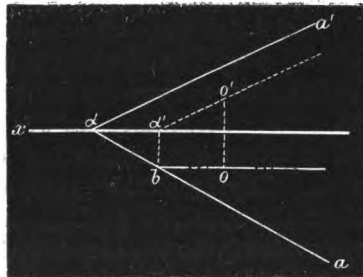
64. *Var. 4°.* The construction might again be modified, by supposing the horizontal trace of the auxiliary plane to be parallel to the ground line. In this case, the intersection of the two planes will be parallel to the vertical trace of the given plane, and so will its vertical projection; and  $(o, o')$  is the required point.



65. In *Variation 2°*, we may consider the point  $o$  as the horizontal projection of a point in a given plane ( $a a'$ ), and it is evident that the point in the plane must be at the intersection with this plane of a perpendicular to the horizontal plane of projection at  $o$ . The preceding construction enables us, therefore, to resolve the general problem which may be stated as follows:

*Having given one of the projections of a point situated in a given plane, find the other projection.*

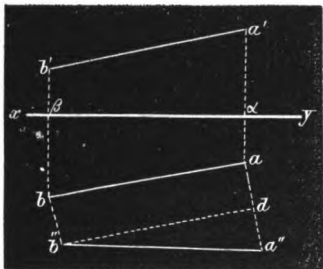
If the vertical projection  $o'$  of a point in the plane ( $a a'$ ) be known, we may determine the horizontal projection, by observing that the point, whose horizontal projection is required, is the point in which a perpendicular to the vertical plane through  $o'$  meets the given plane. If we draw an auxiliary plane through this perpendicular, the vertical trace of which is  $o' a'$ , parallel to  $a' a$ , the vertical trace of the given plane; the horizontal trace will be  $a' b$  perpendicular to the ground line (38-5); and  $b o$ , parallel to the ground line, will be the horizontal projection of the intersection of the auxiliary and given planes. The point  $o$ , at the intersection of the perpendicular through  $o'$  to the ground line (34) with the line  $b o$ , will be the horizontal projection required.



## PROBLEM VI.

66. *Two points being given by their projections, find the true length of the right line joining them.*

Let  $(a, a')$   $(b, b')$  be the two given points. The projections of the right line joining these points will be  $ab$  and  $a'b'$  (46). It is required to find the true length of the line  $(a, a')$   $(b, b')$ .



To do this, we remark, that  $a'\alpha$  and  $b'\beta$  represent the heights of the given points above the horizontal plane (36). If, therefore, we suppose two perpendiculars to be erected to the horizontal plane

at  $a$  and  $b$ , equal respectively to  $a'\alpha$  and  $b'\beta$ , the upper extremities of these perpendiculars will correspond with the given points in space, and the right line connecting them will be the line, the true length of which we wish to determine.

Let the horizontal projecting plane of this line be revolved about its horizontal trace  $ab$ , as an axis, until it coincides with the horizontal plane of projection. The perpendiculars at  $a$  and  $b$ , being perpendicular to the axis before, will also be perpendicular after, the revolution; and the given points in space will fall in the perpendiculars  $bb''$  and  $aa''$ , and at distances from  $ab$ , equal to their respective distances above the horizontal plane. Laying off  $bb'' = b'\beta$ , and  $aa'' = a'\alpha$ , we have  $b''$  and  $a''$  for the revolved positions of the given points; and  $a''b''$  will evidently be the true length of the right line joining them.

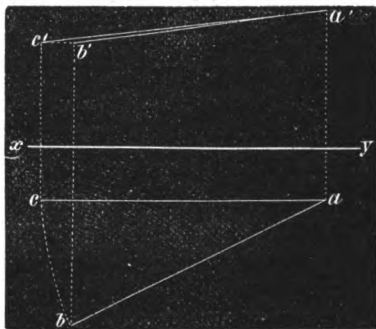
67. If  $b'd$  be drawn parallel to  $ab$ , we shall have a right-angled triangle  $b''da''$ , in which the hypotenuse  $b''a''$  is equal to the true length of the line joining the given points; the base  $db''$  is equal to  $ab$  the horizontal projection of this line; and the altitude  $da''$  is equal to  $a'' - b''$ , that is, to the difference between the perpendicular distances of the given points above the horizontal plane. Hence, *the true length of a right line joining any two points in space, is equal to the hypotenuse of a right-angled triangle, whose base is equal to the horizontal projection of this line, and whose altitude is equal to the difference between the perpen-*

*dicular distances of the given points' above the horizontal plane of projection.*

68. *If a right line be parallel to a plane, its projection on this plane will be equal to itself.* For, in this case, the two projecting perpendiculars of the extremities of the line will be equal, and the given line and its projection will be the opposite sides of a rectangle, and will consequently be equal.

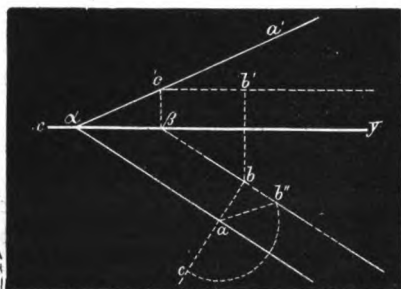
69. *If a right line be not parallel to the plane of projection, its projection will always be less than the line itself.* For, in this case, the true length of the line is the hypotenuse of a right-angled triangle, the base of which is the projection of the line (67); and the base is always less than the hypotenuse.

70. Since the projection of a right line on a plane to which it is parallel, is always equal to the true length of the line (68); we may use this principle to determine the length of a line joining two points in space. For this purpose, revolve the horizontal projecting plane of the line about the vertical line through  $a$ , as an axis, until it becomes parallel to the vertical plane of projection. In this position, the projections  $a$  and  $a'$  will remain unchanged. The point  $b$  will be found at  $c$ , on the line  $ac$  parallel to  $xy$ ,  $ac$  being equal to  $ab$ . Since the point, of which  $b$  is the horizontal projection, remains at the same distance above the horizontal plane; its vertical projection will be found at  $c'$ , at the intersection of the perpendicular through  $c$  with the parallel to the ground line through  $b'$ ; and  $a'c'$  will be the vertical projection of the line joining the two given points, when it is revolved to be parallel to the vertical plane of projection, and is therefore equal to the true length of the line (68).



#### PROBLEM VII.

71. *A plane being given by its traces, find the position of a point situated in this plane, when the plane has been revolved to coincide with either plane of projection.*



Let  $(aa')$  be the given plane. The projections of the point situated in this plane cannot be assumed; for, the projecting perpendiculars which determine the point, must intersect each other in the given plane. But if we assume one of the projections,

which we may always do, we may readily determine the other projection by the construction given in (65).

Let  $b$  be the assumed horizontal projection of a point in the plane  $(aa')$ ;  $b'c'$  will be the vertical projection of the intersection of a plane  $(b\beta c')$ , drawn through the horizontal projecting line of the point, with the plane  $(aa')$ ; and  $b'$ , at the intersection of the perpendicular to the ground line through  $b$  with the vertical projection  $b'c'$ , will be the vertical projection of the point in the given plane.

It is now required to find the position of the point  $(b, b')$ , when the plane  $(aa')$  is revolved about its horizontal trace  $aa$  to coincide with the horizontal plane.

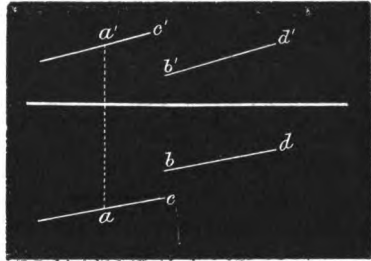
In this revolution, the point  $(b, b')$  preserves its relative position to the axis; and will be found in a perpendicular  $bc$  to the axis  $aa$ , and at a distance from  $aa$ , equal to its distance from it before the revolution. This distance is evidently equal to the hypotenuse of a right-angled triangle, of which the base  $ab$  is equal to the distance of the horizontal projection of the point from the axis, and the altitude equal to the height of the point above the horizontal plane. Laying off  $bb'' = \beta c'$ , we have  $ab''$  for this hypotenuse; and laying off on the perpendicular  $bc$ , the distance  $ac = ab''$ ; the point  $c$  will be the revolved position of the point  $(b, b')$ .

72. If the given plane had been perpendicular to the horizontal plane, the horizontal projection of the point would be found in the horizontal trace of the plane; and in the revolution, the given point would fall at a distance from the axis, equal to its height above the horizontal plane.

**PROBLEM VIII.**

73. *Through a given point draw a right line parallel to a given right line.*

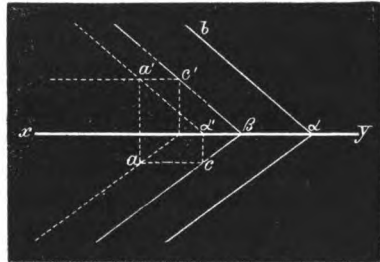
Let  $(a, a')$  be the given point, and  $(b d, b' d')$  the given line. If the two lines be parallel, their projections will also be parallel; for the projecting planes of the lines are respectively parallel to each other (Int. Prop. XII.); and the intersection of these parallel planes with the planes of projection will also be parallel. (Int. Prop. XIX.) Hence,  $a c$  parallel to  $b d$ , and  $a' c'$  parallel to  $b' d'$ , will be the projections of the parallel line.



**PROBLEM IX.**

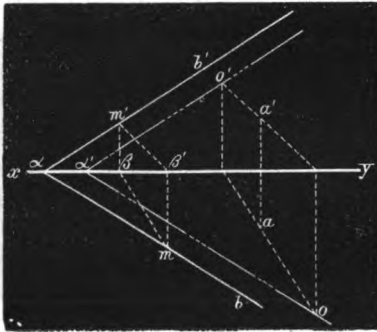
74. *Draw a plane through a given point parallel to a given plane.*

Let  $(a, a')$  be the given point, and  $(b \alpha b')$  the given plane. Since the planes are required to be parallel, their traces will be respectively parallel; we know then the direction of the traces of the required plane. Through the given point  $(a, a')$ , and in the plane of the required plane, let a line be drawn parallel to the horizontal trace of this plane. Its horizontal projection will pass through  $a$  and be parallel to  $b \alpha$ ; and its vertical projection will pass through  $a'$  and be parallel to the ground line (38-2). This line meets the vertical plane of projection at  $c'$ . The point  $c'$  is one point in the vertical trace of the required plane. Drawing  $c' \beta$  parallel to  $b' \alpha$ , we have the vertical trace of this plane. A line through  $\beta$ , parallel to the horizontal trace of the given plane, will give  $\beta c$  as the horizontal trace of the required parallel plane.



75. We might determine another point in the horizontal trace, and thus verify the above construction: by drawing, through the given point, a line parallel to the vertical trace of the required plane. Its vertical projection would pass through  $a'$ , and would be parallel to  $c'\beta$ :  $a'a'$  is this projection, and  $ac$ , parallel to the ground line, is the horizontal projection of the auxiliary line. The line  $(a'a', ac)$  meets the horizontal plane of projection at  $c$ : the point  $c$  is therefore a point in the horizontal trace of the required plane. If the construction be accurately made, the point  $c$  will be in the trace  $\beta c$  before found.

76. This problem may be solved in a more general way, by drawing in the given plane any right line whatever; and then drawing, through the given point, a line parallel to this line. The parallel line will be a line of the required plane; and the points in which it meets the planes of projection will determine a point in each trace of the required plane. The traces may then be drawn parallel, respectively, to the traces of the given plane. We may verify the construction, by observing whether their traces intersect in the ground line.



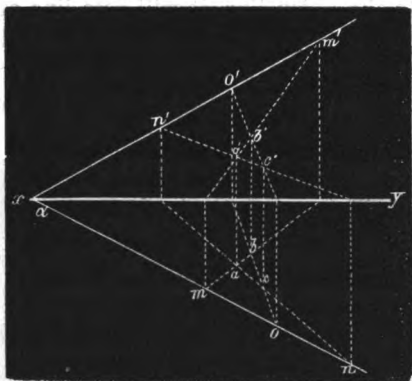
Let  $(a, a')$  be the given point, and  $(b, b')$  be the given plane. Assume any points  $m$  and  $m'$  in the traces of the given plane; and draw the perpendiculars  $m\beta$  and  $m'\beta'$  to the ground line:  $\beta$  will be the vertical projection of the point  $m$ , and  $\beta'$  will be the horizontal projection of the point  $m'$ ; and  $m\beta$  and  $m'\beta'$  will be the projections of a

line situated in the plane  $(b, b')$ . Drawing through  $a$  and  $a'$  the lines  $ao$  and  $a'o'$ , parallel respectively to  $m\beta$  and  $m'\beta'$ , they will be the projections of a line passing through the given point and parallel to the line assumed in the given plane. The points  $o$  and  $o'$ , in which this line meets the planes of projection, determine a point in each trace of the parallel plane; and  $oa'$  and  $o'a'$  parallel respectively to  $ba$  and  $b'a'$  will be the traces of this plane. The point of intersection of these traces,  $a'$ , must be in the ground line.

**PROBLEM X.**

77. *Draw a plane through three given points.*

Let  $(a, a')$ ,  $(b, b')$ , and  $(c, c')$  be the given points. If these points be connected, two and two, by right lines; these lines will be lines of the required plane, and their projections will pass through the corresponding projections of the given points. If we determine the points in which these lines meet the planes of projection, we shall have points in the traces of the required plane.



The line  $(ab, a'b')$  meets the vertical plane at  $m'$ , and the horizontal plane at  $m$ ; the line  $(ac, a'c')$  meets the vertical plane at  $n'$ , and the horizontal plane at  $n$ ; and the line  $(bc, b'c')$  meets the planes of projection at  $o$  and  $o'$ . The points  $m, n, o$ , are points in the horizontal trace of the required plane; and the points  $m', n', o'$ , are points in its vertical trace. The lines  $nom$  and  $m'o'n'$  are therefore the traces of the required plane. These traces must be right lines, and must intersect the ground line at  $a$ .

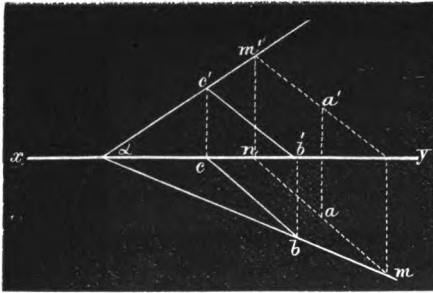
78. *Var. 1°. If the line which connects two of the given points be parallel to either plane of projection, the vertical for example, then the point in which it meets this plane will be at an infinite distance, and cannot therefore be used in the construction. In this case, the vertical trace of the required plane must evidently be parallel to this line, and also to its vertical projection. (Int. Prop. XII.)*

79. *Var. 2°. If the line which connects two of the given points be parallel to both planes of projection, it will meet neither of these planes, and the traces of the required plane being parallel to the projections of this line, will be parallel to the ground line. (Int. Prop. XI.)*

**PROBLEM XI.**

80. *Draw a plane through a given point and through a given right line.*

If a right line be drawn through the given point and parallel to the given line, it will be a line of the required plane. If we determine the points in which this line and the given line meet the planes of projection, we shall have two points in each trace of the required plane, and the plane is therefore determined.



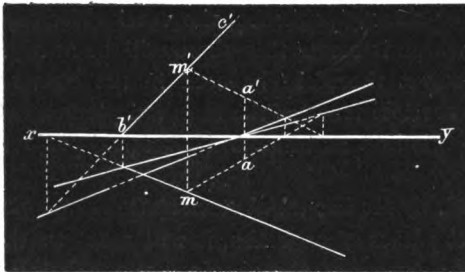
Let  $(a, a')$  be the given point, and  $(bc, b'c')$  the given line. We have  $(am, a'm')$  for the line through  $(a, a')$  parallel to  $(bc, b'c')$  (73); and this line meets the planes of projection at  $m$  and  $m'$ . The given line meets the planes of projection at  $b$  and  $b'$ , hence

$m$  and  $m'$  are the traces of the required plane.

If the construction be accurately made, these traces must intersect each other in the ground line at  $a$ .

81. **Variation.** Instead of using a parallel to the given line, we might have used the line connecting the given point with any point whatever of the given line.

To fix the position of the assumed point in the line  $(bc, b'c')$ , we must make its projections  $m, m'$  fulfil the condition of being in the same perpendicular to the ground line (34). We have



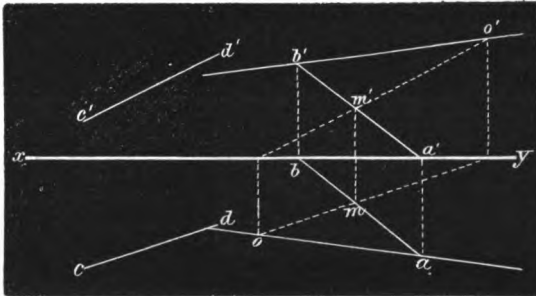
$(am, a'm')$  for the line joining the point  $(a, a')$  and the point  $(m, m')$ . It will be a line of the required plane, and the traces of the plane may be readily determined, as before.

**PROBLEM XII.**

82. *Through a given right line draw a plane parallel to a given right line.*

If any point be assumed on the line through which the plane is to be drawn, and a line be drawn through it parallel to the other line, the plane through this auxiliary line and the first line will be the required parallel plane.

Let it be required to draw a plane through the line  $(ab, a'b')$  parallel to the line  $(cd, c'd')$ . We have  $(mo, m'o')$  for the parallel line through the point  $(m, m')$ . If we determine the points

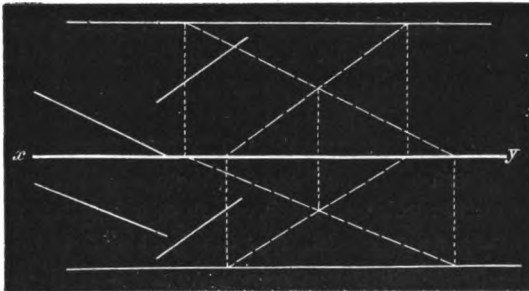


in which the lines  $(mo, m'o')$  and  $(ab, a'b')$  meet the planes of projection, we shall have two points in each trace of the required plane. The traces are readily determined.

**PROBLEM XIII.**

83. *Draw a plane through a given point and parallel to two given right lines.*

Through the given point draw two right lines parallel to the



two given lines. The plane of these lines will pass through the given point and be parallel to the two given lines.

The construction may readily be made as in the figure.

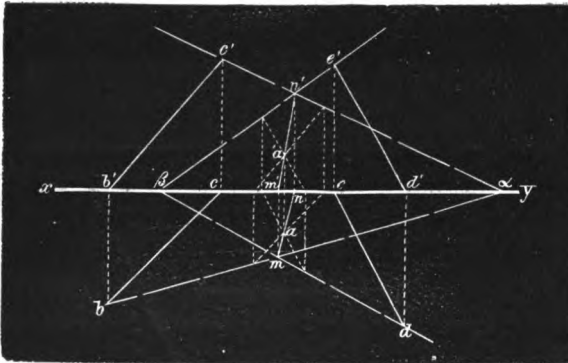
#### PROBLEM XIV.

84. Find a right line that shall pass through a given point and meet two given right lines.

**First Solution.** Draw a plane through the given point and one of the given lines as in **Prob. XI.**; and another plane through the given point and the other given line. The line of intersection of these two planes, determined by **Prob. IV.**, will be the required line.

Let  $(a, a')$  be the given point;  $(bc, b'c')$   $(de, d'e')$  the given lines.

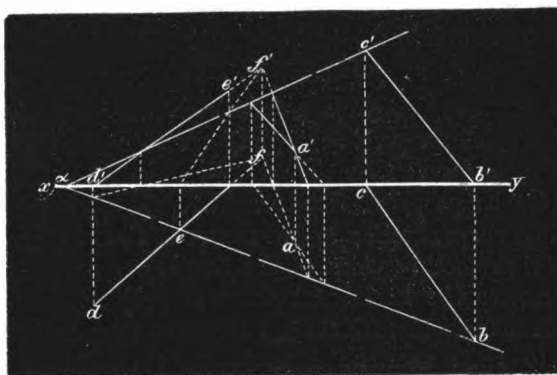
The plane through  $(a, a')$  and  $(bc, b'c')$  has  $c' \propto$  and  $b \propto$  for its



traces; and  $\beta e'$  and  $\beta d$  are the traces of the plane through  $(a, a')$  and the line  $(de, d'e')$ . The intersection of the planes  $(b \propto c')$  and  $(d \propto e')$  is the line  $(mn, m'n')$ , which is the required line.

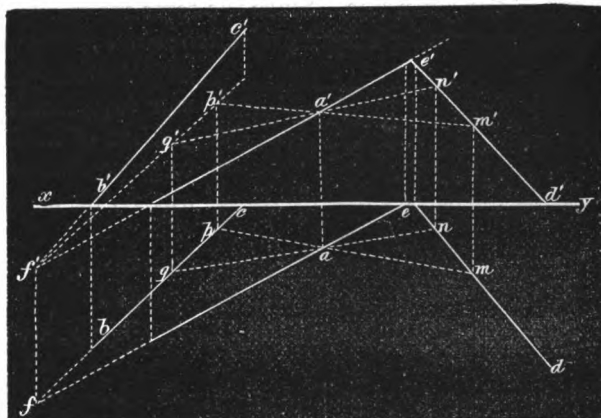
85. **Second Solution to Prob. XIV.** If we pass a plane through the given point and one of the given lines as in **Prob. XI.**, and find the point of intersection of the other given line with this plane, as in **Prob. V.**; and join this point of intersection with the given point, we shall have a right line which will, in general, meet the first line.

The plane through  $(a, a')$  and  $(bc, b'c')$  is  $(c'a'b)$ . The line



$(de, d'e')$  meets this plane in the point  $(f, f')$ . The right line joining this point with the given point is the required line.

86. **Third Solution to Prob. XIV.** Take, at will, any



two points  $(m, m')$   $(n, n')$  on the line  $(de, d'e')$ . Through these points and the point  $(a, a')$  draw two lines, and determine the points  $(p, p')$   $(q, q')$  in which they meet the horizontal projecting plane of the line  $(bc, b'c')$  as in **Prob. V**. The line  $(bc, b'c')$  will meet the line  $(pq, p'q')$  which joins the points  $(p, p')$   $(q, q')$  in a point  $(f, f')$ ; and if we join the point  $(f, f')$  with the given point  $(a, a')$  by a right line, this line will be the required line.

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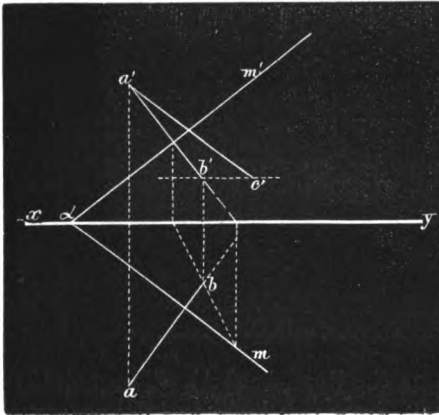
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For it passes through the given point, is in the plane of the first line, and must therefore, in general, meet it, and it also meets the second line.

### PROBLEM XV.

87. *Draw through a given point a right line perpendicular to a given plane, find the point in which it meets the plane, and the true length of the perpendicular.*

If a line be perpendicular to a plane, the projections of the line will be perpendicular to the traces of the plane. For, the horizontal projecting plane of the line is perpendicular, at the same time, to the horizontal plane of projection and to the given plane (Int. Prop. XXI.); and must therefore be perpendicular to their common intersection (Int. Prop. XXVIII. Cor. 2), which is the horizontal trace of the given plane. But the horizontal trace of the given plane being perpendicular to the horizontal projecting plane of the line, is also perpendicular to the horizontal trace of this plane, which is the horizontal projection of the given line. Since the same reasoning equally applies to the vertical plane of projection; we conclude, that if



a right line be perpendicular to a plane, the projections of the line will be perpendicular to the traces of the plane. This principle being established, the Problem is readily solved.

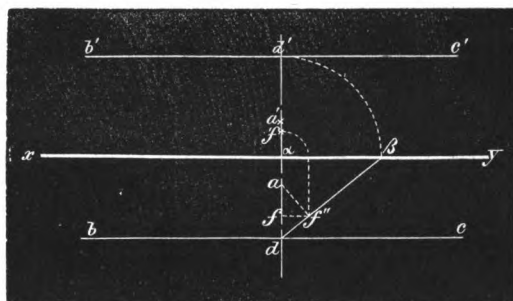
Let  $(a, a')$  be the given point, and  $(m, m')$  the given plane. The lines  $ab$  and  $a'b'$  perpendicular to the traces  $m$  and  $m'$ , are the projections of the required

perpendicular. By Prob. V. we determine  $(b, b')$  as the point in which this perpendicular meets the plane  $(m, m')$ ; and by Prob. VI. we find  $a'b'$  to be the true length of this perpendicular.

88. **Variation.** *Let the traces of the given plane be parallel to the ground line.*

In this case, the construction given above would fail, and it would be necessary to use an auxiliary plane as in **Prob. IV. Var. 3d.**

Let  $(a, a')$  be the given point, and  $bc$  and  $b'c'$  the traces of the given plane. Through  $(a, a')$  draw a plane perpendicular to the ground line;  $aa$  and  $a'a$  are its traces. This plane intersects the given plane in a line which meets the vertical plane of projection at  $d'$  and the horizontal plane at  $d$ . If the auxiliary plane be revolved about its horizontal trace  $aa$  until it coincides with the horizontal plane; the point  $d'$  will fall at  $\beta$ ,  $a\beta$  being equal to  $a'd'$ . The point  $d$  being in the axis will remain fixed, and  $d\beta$  will be the revolved position of the intersection of the auxiliary and given planes. But the perpendicular line through  $(a, a')$  to the given plane, being in the vertical plane through  $(a, a')$ , is perpendicular to the intersection of this plane with the given plane; and in the revolution it remains perpen-



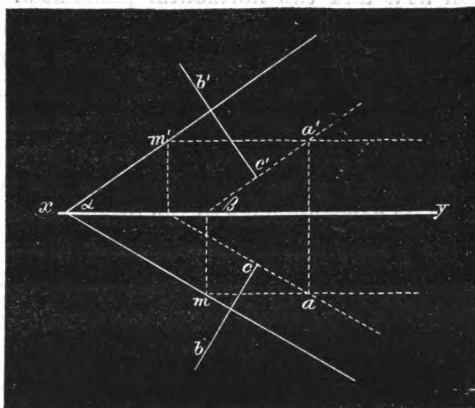
dicular. Hence,  $af''$ , perpendicular to  $d\beta$ , will be the revolved position of this line; and  $f''$  is the revolved position of the point in which the perpendicular meets the given plane. In the counter-revolution, the point  $f''$  will be horizontally projected at  $f$ , and vertically at  $f'$ ;  $f'a$  being equal to  $f'f''$ . We have  $(f, f'')$  for the point in which the perpendicular meets the given plane; and  $af''$  for the true length of the perpendicular.

The projections of the perpendicular are evidently in the traces  $aa$  and  $a'a$  (7).

### PROBLEM XVI.

89. Through a given point draw a plane perpendicular to a given right line.

Let  $(a, a')$  be the given point, and  $(bc, b'c')$  the given line. Since the required plane has to be perpendicular to the given line, the traces of the plane will be perpendicular to the projections of the line (87). We know then the *direction* of the required traces. Conceive a right line to be drawn through  $(a, a')$  parallel to the horizontal trace of the required plane; this line will be a line of the required plane. Its horizontal projection



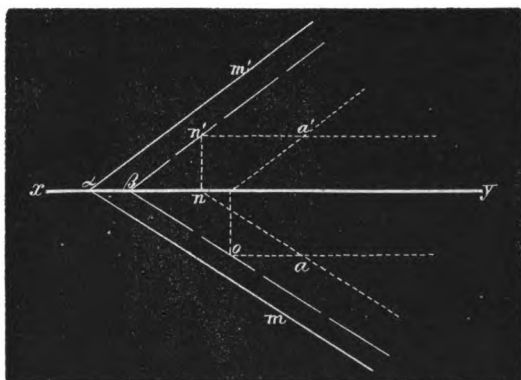
will pass through  $a$ , and will be parallel to the horizontal trace of the required plane (72); that is, will be perpendicular to  $bc$ . Its vertical projection will pass through  $a'$ , and be parallel to the ground line (38-2). This line meets the vertical plane of projection at  $m'$ . This point is a point in the vertical trace of the required plane, and  $m'a$ , perpendicular to  $b'c'$ , will be the vertical trace of this plane; and  $am$ , perpendicular to  $bc$ , will be its horizontal trace.

90. We might verify the horizontal trace, by finding another point of it. For, if we draw through  $(a, a')$  a line parallel to the vertical trace of the required plane, the line  $a'\beta$ , perpendicular to  $b'c'$  or parallel to  $am'$ , will be its vertical projection; and the line  $am$  parallel to the ground line will be its horizontal projection. If the construction be accurately made, the point  $m$ , in which this line meets the horizontal plane, will be in the trace  $am$ .

#### PROBLEM XVII.

91. Draw through a given point a plane parallel to a given plane.

Let  $(a, a')$  be the given point ;  $(m \alpha m')$  the given plane. Since the planes are to be parallel, their traces will be parallel, (Int.



Prop. XIX.). We know then the *direction* of the traces of the required plane. If a line be drawn through  $(a, a')$  parallel to the horizontal trace of the required plane, its horizontal projection will pass through  $a$ , and be parallel to the horizontal trace of the required plane; that is, it will be parallel to  $\alpha m$ ; and its vertical projection will pass through  $a'$ , and will be parallel to the ground line. The line  $(a n, a' n')$  meets the vertical plane of projection at  $n'$ ;  $n' \beta$  parallel to  $\alpha m'$ , and  $\beta o$  parallel to  $\alpha m$ , will be the traces of the required plane. By drawing a line through the point  $(a, a')$  parallel to the vertical trace of the required plane, we may determine another point in the horizontal trace, and thus verify the construction.

92. *Second Solution.* If, from any point of the ground line, two lines be drawn perpendicular to the projections of the given line, respectively, they will be the traces of a plane perpendicular to the given plane, and passing through the assumed point on the ground line. If now a plane be drawn through the given point  $(a, a')$ , parallel to this plane, it will be the required plane.

### PROBLEM XVIII.

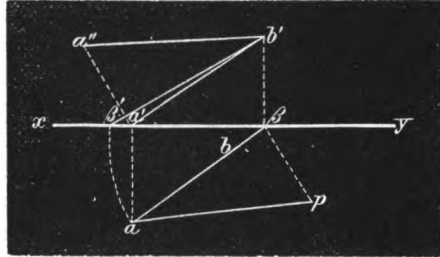
93. *Through a given point draw a right line perpendicular to a given right line.*

Let  $(a, a')$  be the given point, and  $(b c, b' c')$  the given line.



the plane is measured by the angle which this line makes with its projection on this plane.

Let  $(ab, a'b')$  be the given line oblique to the planes of projection. Let it be required to determine the angle which it makes with the horizontal plane. The line  $(ab, a'b')$  meets the planes of projection at  $a$  and  $b'$ . Revolve the horizontal projecting plane of this line about the horizontal trace  $a\beta$  until it coincides with the horizontal plane. The point  $b'$  falls at  $p$  in a perpendicular to the axis, and at a distance from it equal to  $\beta b'$ . The point  $a$ , being in the axis, remains fixed. The line  $ap$  is the revolved position of the given line; and the angle  $\beta ap$  is the angle which the given line makes with the horizontal projection  $ab$ , which angle is the measure of the angle required.



96. The angle which the line makes with the horizontal plane might be determined, by revolving the plane  $(a\beta b')$  about its vertical trace  $b'\beta$ , as an axis, until it coincides with the vertical plane. The point  $a$  is found at  $\beta'$ ,  $\beta\beta'$  being equal to  $a\beta$ ; and  $\beta'b'$  will be the revolved position of the line, and  $\beta\beta'b'$  the required angle.

97. The angle which the line makes with the vertical plane of projection is determined by an analogous process, and is  $a'b'a''$ .

### PROBLEM XX.

98. *A plane being given by its traces, find the angle which it makes with either of the planes of projection, say, the horizontal plane.*

In general, the angle which two planes make with each other, is measured by the angle formed by two right lines, one in each plane, and perpendicular to the common intersection of the planes at the same point (Int. Def. 10).

Let  $a\alpha$  and  $a'\alpha$  be the traces of the given plane.

If a plane be drawn perpendicular to the horizontal trace  $a\alpha$ , it will be perpendicular to the given plane, and also to the hori-



101. We may readily determine the angle between the traces of the given plane, by revolving the plane about its horizontal trace, and find the revolved position of the vertical trace. The point  $a'$  falls at  $a'''$  in a perpendicular to the axis, and at a distance from it equal to the true length of the line joining the points  $a$  and  $a'$ , that is, equal to  $a a''$ . The point  $a$  remains fixed, and  $a''' a$  is the revolved position of the trace  $a a'$ , and  $a''' a a$  is the angle of the traces.

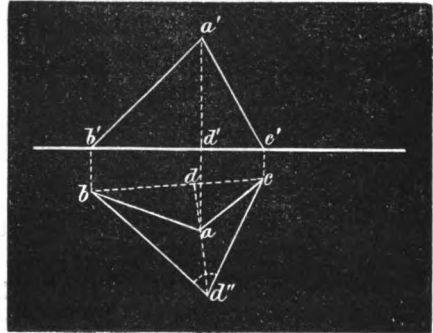
### PROBLEM XXI.

102. Find the angle included between two given lines.

Let  $(a b, a' b')$ ,  $(a c, a' c')$  be the two given lines.

If the lines intersect, the projections  $a$  and  $a'$  of their point of intersection must fulfil the condition of being in the same perpendicular to the ground line. The given lines meet the horizontal plane of projection at  $b$  and  $c$ , and  $b c$  is the horizontal trace of the plane of these lines.

The line  $b c$  is the base of a triangle, the vertex of which is projected at  $(a, a')$ . The angle opposite to the base is the required angle. If  $a d$  be drawn perpendicular to  $b c$ , and from the point  $d$  a line be drawn to the vertex  $(a, a')$ , this line will lie in a vertical plane through  $(a, a')$  perpendicular to



$b c$ , and will also be perpendicular to  $b c$ ; so that, if the plane of the triangle be revolved about  $b c$ , as an axis, until it coincides with the horizontal plane, this perpendicular will take the direction  $d d''$ , and the point  $(a, a')$  will be found at  $d''$ , at a distance from  $b c$ , equal to the hypotenuse of a triangle of which  $a d$  is the base and  $a' d'$  the altitude. Joining  $d''$  with  $b$  and  $c$ , the angle  $b d'' c$  is the angle required.

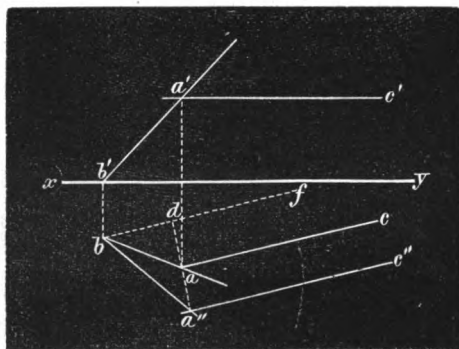
103. The vertex  $d''$  might also be found by finding the true length of the sides of the angle; the projections of these sides being, respectively,  $a b$  and  $a' b'$ ,  $b c$  and  $b' c'$ , as in **Prob. VII.**

Then describing two arcs from  $b$  and  $c$ , as centres, with radii equal to the true length of the sides, their intersection determines the point  $d''$ .

104. If the given lines be not in the same plane, the angle included between them is measured by the angle included between two right lines drawn through any point and parallel to the two given lines. The construction can readily be made.

105. *Var. 1°.* If one of the given lines be parallel to the horizontal plane, its projection on the vertical plane is parallel to the ground line.

In this case, the horizontal trace  $bf$  of the plane of the two lines will be parallel to the horizontal projection of the line



which is assumed to be parallel to the horizontal plane. When the plane of the lines is revolved to coincide with the horizontal plane, the horizontal line will revolve parallel to the horizontal projection, and  $ba''c''$  is the angle required.

106. *Var. 2°.* If one of the given lines be parallel to the ground line.

Its two projections will be parallel to the ground line; the traces of the plane of the lines will also be parallel to the ground line. The construction may readily be made.

107. *Var. 3°.* If the vertex of the angle be in the horizontal plane.

In this case, we assume a point on either of the given lines, and draw a line parallel to the other given line. The angle between this line and the first line will be the angle required. The construction is readily made.

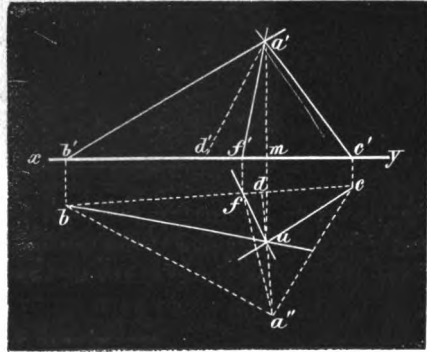
108. **Remark.** If two right lines are perpendicular to each other, and are projected on a plane to which one of them is parallel; their projections will also be perpendicular to each other.

For, if through the line which is parallel to the plane on which the projection is made, a plane be drawn perpendicular to the other line, its trace will be parallel to the line through which the plane is drawn (Int. Prop. XVI.), and also to the projection of this line. But the projection of the line to which the plane is drawn perpendicular, is perpendicular to the trace of this plane (87); it is therefore perpendicular to the projection of the line through which the plane is drawn.

### PROBLEM XXII.

109. Through the point of intersection of two given lines, draw a right line that shall bisect their angle equally.

Let  $(ab, a'b')$  and  $(ac, a'c')$  be the given lines. Construct the angle between the two given lines as in **Prob. XXI.** We have  $ba''c$  for this angle. Bisect this angle equally by the line  $a''f$ . It is now required to determine the projections of the line  $a''f$ , when the plane of the given lines assumes its true position. In the counter-revolution, the vertex  $a''$  falls back in a perpendicular to the axis, and  $(a, a')$  will be the position of this point; the point  $f$  in which the bisecting line meets the axis remains fixed;  $f$  will therefore be its own horizontal projection, and  $f'$ , in a perpendicular to the ground line through  $f$ , will be its vertical projection, and  $(af, a'f')$  will be the projections of the bisecting line.



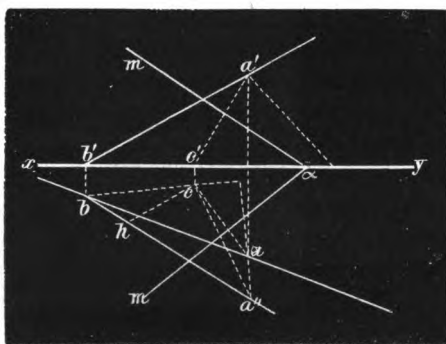
### PROBLEM XXIII.

110. Find the angle which a given right line makes with a plane to which it is not parallel.

Let  $(ab, a'b')$  be the given line, and  $(m \propto m')$  the given plane.

Since the angle which a line makes with a plane, is equal to the angle which the line makes with its projection on that plane (95); if, from any point of the given line, a perpendicular be drawn to the given plane, the angle which the perpendicular makes with the given line, will be the complement of the angle which this line makes with its projection on this plane. The problem is thus reduced to finding the angle between two right lines as in **Prob. XXI.**

Through any point ( $a a'$ ) draw the lines  $ac$  and  $a'c'$  perpendicular to the traces of the given plane; ( $ac, a'c'$ ) will be a line perpendicular to the given plane (87). The horizontal trace of



the plane of the given line and this perpendicular is  $bc$ , and  $ca''b$  will be the angle between these lines. If we draw  $ch$  perpendicular to  $a''b$ ,  $a''ch$  will be the angle required.

#### PROBLEM XXIV.

111. *Find the angle included between two given planes.*

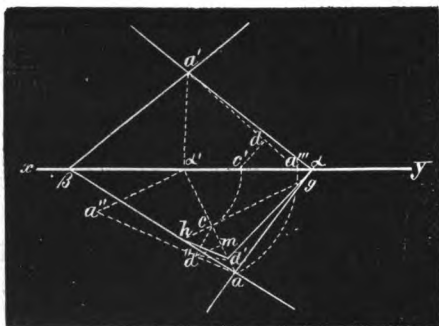
The angle between two planes is measured by the angle included between two right lines drawn, one in each plane, perpendicular to the common intersection of the planes at the same point (Int. Def. 10). If a plane be drawn perpendicular to the intersection of the two given planes, it will intersect the given planes in two right lines, which will make with each other the required angle.

Let ( $\alpha a a'$ ) and ( $\alpha \beta a'$ ) be the two given planes.

These planes intersect each other in a line which meets the vertical plane of projection at  $a'$ , and the horizontal plane at

$a$ , and  $a'$  is the horizontal projection of the intersection. The line  $gch$  perpendicular to  $a \propto'$  is the horizontal trace of a plane perpendicular to the intersection of the two given planes (87). The perpendicular plane intersects the given planes in lines which meet the horizontal plane of projection at  $g$  and  $h$ ; and which form with the line  $gh$  a triangle, the angle opposite to  $gh$  being the angle required.

Let the plane of the triangle be revolved about  $gh$ , as an axis, until it coincides with the horizontal plane. The vertex of the triangle is in the vertical plane  $ac\alpha'$ . But this plane is perpendicular to  $gh$ , since  $gh$  is perpendicular to  $a\alpha'$ . Then, the line which joins  $c$  and the vertex of the triangle, is perpendicular to  $gh$ ; and in the revolution of this plane, this line will take the direction  $ca$ . We have now to find the true length of this line.



The plane of the tri-  
angle is perpendicular to the intersection of the two given  
planes. The line which joins the point  $c$  with the vertex of  
the triangle, being in the plane of the triangle, is also perpen-  
dicular to this intersection. If, therefore, we revolve the hori-  
zontal projecting plane of the intersection of the two given  
planes, about its horizontal trace, until it coincides with the  
horizontal plane, the line of intersection will take the position  
 $a a''$ , since  $a'$  remains fixed, being in the axis, and  $a'$  falls in a  
perpendicular to  $a a'$ , and is found at  $a''$ ,  $a' a''$  being equal to  
 $a' a'$ ;  $c d''$ , perpendicular to this revolved line, is the revolved  
position of the line joining the point  $c$  with the vertex of the  
triangle, and will be the true distance of the vertex from the  
base. If we lay off  $c d' = c d''$ , and draw  $d' g$  and  $d' h$ ;  $g d' h$  will  
be the angle required.

112. We might have determined the true length of the line joining the vertex of the triangle with the base, by revolving the horizontal projecting plane of the intersection of the two

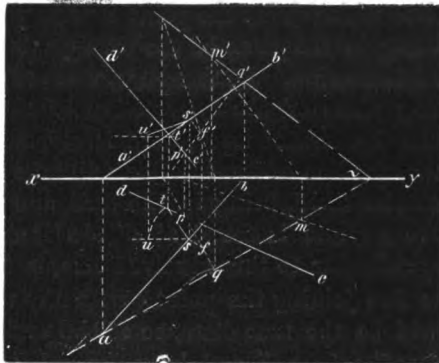
given planes about its vertical trace, until it coincides with the vertical plane of projection. In this position,  $a'a'''$  would be the revolved position of the intersection of the planes;  $c'$  the revolved position of the point  $c$ ; and  $c'd$  the true length of the perpendicular. The vertex of the triangle is evidently horizontally projected at  $m$ , where a perpendicular from  $d''$  to  $a'a'$  meets this line.

### PROBLEM XXV.

113. Find the shortest distances between two right lines, not in the same plane.

If a plane be drawn through one of the lines parallel to the other line, every point of the parallel line will be equally distant from the plane drawn parallel to it; and since the shortest distance required must join a point of this line with a point of the parallel plane; it follows, that this distance cannot be less than the perpendicular distance between the line and plane. This perpendicular is also the shortest distance required. (Smith's Legendre, Book V. Prop. XXI.)

Let  $(ab, a'b')$ ,  $(cd, c'd')$  be the given lines: let a plane be drawn through  $(ab, a'b')$  parallel to  $(cd, c'd')$ , as in **Prob. XII**. We have  $am, a'm$  for its traces. Through any point



$(p, p')$  of the line  $(cd, c'd')$  draw a perpendicular to this plane as in **Prob. XV**. The projections of this line are perpendicular to the traces of the plane, and  $(pq, p'q')$  will be the perpendicular line. The perpendicular meets the plane  $(m, m')$  at

the point  $(f, f')$ . If a line be drawn through  $(f, f')$  parallel to  $(cd, c'd')$ , the projections will be respectively parallel to  $(cd, c'd')$ , and we have  $(fs, f's')$  for the parallel. These projections meet those of the line  $(ab, a'b')$  at the point  $(s, s')$ . The point  $(s, s')$  is therefore the intersection of the parallel line  $(fs, f's')$  with the line  $(ab, a'b')$ . If now we draw through  $(s, s')$ , the lines  $st, s't'$ , parallel to  $pq, p'q'$ , and terminated by the projections  $cd$  and  $c'd'$ ; we shall have the projections of the shortest lines between the two given lines. The projections  $st, s't'$  are evidently perpendicular to the traces  $ma$  and  $m'a$  (87.)

The true length of the shortest distance may be readily constructed by **Prob. XV.** 114. If the two given lines intersected, then the plane through one line parallel to the other line, would be the plane of the lines; and the perpendicular which measures the shortest distance reduces to zero.

115. If the two lines be parallel, then the **right** line drawn through one of the lines parallel to the other line, will coincide with the first line; the parallel plane becomes indeterminate, and the above construction fails. In this case, the shortest distance will evidently be any perpendicular drawn from one line to the other.

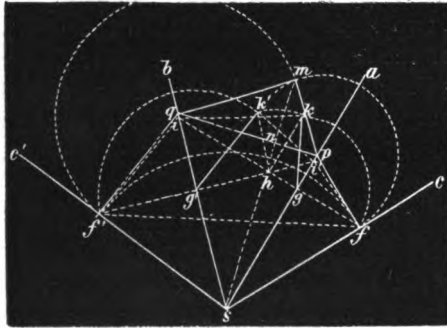
### PROBLEM XXVI

116. *Having given the three faces of a trihedral angle, find the angles of inclination which the faces make with each other, two and two; that is, find the dihedral angles.*

In a trihedral angle there are three faces and three dihedral angles. If we propose to determine three of these parts, when the other three are known; there may be six cases as follows: 1°. When the three faces are known. 2°. Two faces and their dihedral angles known. 3°. Two faces and the dihedral angle opposite one of them. 4°. One face and the adjacent dihedral angles. 5°. One face, one adjacent dihedral angle, and one dihedral angle opposite the face. 6°. Three dihedral angles. The three last cases may be reduced to the solution of the three first. It must be remembered, however, that two conditions must always be established when a trihedral angle is formed with three given faces. 1st. The sum of the plane angles must

be less than four right angles (Smith's Leg., Book V. Prop. XXXVI.). 2d. The greatest plane angle must be less than the sum of the other two (Smith's Leg., Book V. Prop. XXXV.).

To determine the three diedral angles, when the three faces of the triedral angle are known; in any plane taken as the horizontal plane of projection, make the angle  $asb$  equal to the plane angle of one of the faces; and let the planes of the other faces be revolved about  $as$  and  $sb$ , as axes, until they coincide with the horizontal plane. In this position, draw  $asc$  and  $bsc'$  equal to the two other angles of the plane faces. It is evident we may construct the triedral angle by revolving the planes  $asc$ ,  $bsc'$ , around  $as$  and  $sb$ , until  $sc$  and  $s'c'$  coincide. Take, on these sides,  $sf = sf'$ , and draw  $fgh$  and  $f'g'h$  perpendicular to  $sa$  and  $sb$ , and intersecting each other in  $h$ . During the revolution of the two faces, the points  $f$  and  $f'$  describe, about  $g$



and  $g'$  as centres, circles situated in the vertical planes whose traces are  $gh$  and  $g'h'$ ; and when the sides  $sc$  and  $s'c'$  coincide, the points  $f$  and  $f'$  unite and form one point. The lines  $fg$  and  $f'g'$  form with  $gh$  and  $g'h'$  angles, which are equal to the angles made by the lateral faces with the horizontal plane  $asb$ .

To determine these angles; revolve the planes of these angles around the traces  $fg$  and  $f'g'h$  until they coincide with the horizontal plane. The common line of intersection of these two planes corresponds with the vertical line at the point  $h$ ; and when the revolution is made around  $gh$ , coincides with  $hk$  perpendicular to  $gh$ ; and when it is made around  $g'h$ , it falls in  $h'k'$  perpendicular to  $g'h$ ; and the point of union of the points  $f$  and

$f'$  in space, will fall on these perpendiculars, and at distances from  $gh$  and  $g'h$ , equal respectively to  $gf$  and  $g'f'$ , and will therefore be found at  $k$  and  $k'$ : hence, drawing  $gk$  and  $g'k'$ , the angle  $kg'h$  will be the inclination of the faces  $asc$  and  $asb$ , and the angle  $k'g'h$  will be the inclination of the faces  $bsc'$  and  $asb$ .

We might determine the third angle of inclination, by making on one of the faces  $asc$ ,  $bsc'$ , the same construction which has been made on the face  $asb$ ; but it may be more readily determined as follows. Conceive a plane to be drawn through the points in space corresponding to  $f$  and  $f'$ , perpendicular to the third edge: it will intersect the two lateral faces in two right lines, which will make with each other the required angle of inclination. One of these lines revolves on  $fp$  perpendicular to  $sc$ , the other, on  $f'q$  perpendicular to  $sc'$ . It is evident that the points  $p$  and  $q$ , in which these lines meet the edges  $as$  and  $bs$ , have not changed in position during the revolution; hence, drawing  $pq$ , we shall have on the face  $asb$ , the trace of a plane which contains the required angle; and if we revolve this plane around  $pq$ , the vertex of this angle will be found at the intersection  $m$  of the arcs described with the radii  $pf$  and  $qf'$ : therefore, the angle  $pmq$  is equal to the third angle required.

117. Many verifications may be noted.

1°. The lines  $hk$  and  $hk'$  must be equal, as revolutions of the same vertical.

2°. The line  $sh$  being the projection of the third edge on the plane  $asb$ , and  $pq$  being the trace of a plane perpendicular to this edge, it follows, that  $sh$  must be perpendicular to  $pq$ . If  $sh$  is perpendicular to  $pq$ , the right line drawn from the point in space corresponding to  $f, f'$ , to the point  $n$  in which  $sh$  meets  $pq$ , must also be perpendicular to  $pq$ : hence, in the revolution of  $pqm$ , this line must be found on the prolongation of  $sn$ ; and the point  $m$  must be on this prolongation. Thus, the line  $sh$  is perpendicular to  $pq$ , and passes through  $m$ .

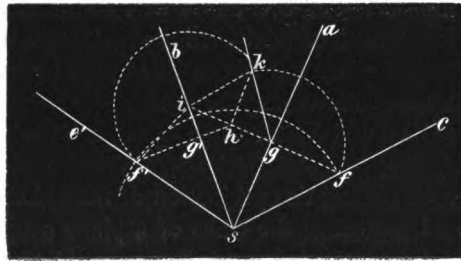
3°. If we produce  $gh$  until it intersects  $sb$  at  $i$ , the distances  $if'$  and  $ik$ , must be equal. In like manner, if we produce  $g'h$  until it meets  $sa$  in  $i'$ , we must also have  $i'f = i'k$ .

**PROBLEM XXVII.**

118. *Knowing two faces of a triedral angle, and their inclination, find the third face, and the two other angles of inclination of these faces with the third face.*

Let  $asb$  and  $asc$  be the two given faces, when one is revolved around the common edge to coincide with the plane of the other. If we draw  $fgh$  perpendicular to  $as$ , the lines  $gh$  and  $gf$  will be the intersections of these faces with the plane drawn perpendicular to the edge  $as$  at the point  $g$ ; and when the two faces are in their true position, these lines make with each other an angle which is equal to the given inclination.

Let the plane of this angle be revolved around  $gh$  until it coincides with the plane  $asb$ ; the line  $gf$  will take the direction  $gk$ , and the angle  $kg'h$  will be equal to the dihedral angle of the faces. Further, the distance  $gf$  will not vary during the revolution: hence, if we take  $gk = gf$ , and draw  $kh$  perpendicular to  $gh$ , the point  $h$  will be the foot of the perpendicular let fall



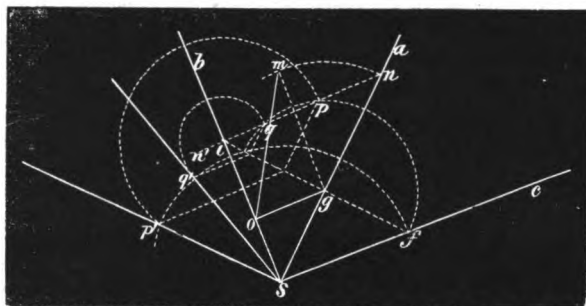
on the plane  $asb$ , through the point on the third edge corresponding with  $f$  and  $k$ . Let this third edge be revolved around  $bs$  until it coincides with the plane  $asb$ ; it is evident that the point of this edge corresponding to  $f$  will fall somewhere on the perpendicular  $hg'f'$  to  $bs$ , and since it must be at the same distance from the vertex  $s$ , as the point  $f$ , it must be found at  $f'$ , and hence  $bsf'$  is the third face. Knowing the three faces, the dihedral angles are determined by the preceding problem.

If we produce  $gh$  to  $i$ ,  $if' = ik$ .

**PROBLEM XXVIII.**

119. *Having two faces of a triedral angle, and the dihedral angle opposite one of them, find the third face and the two other dihedral angles.*

Let  $asb$  and  $asc$  be the two given faces, and suppose that the dihedral angle opposite to the face  $asc$  is also known. If the



triedral angle in space be intersected by a plane perpendicular to the edge  $as$ , and we revolve the triangle which results from the intersection of this plane with faces of the triedral angle, until it coincides with the plane  $asb$ ;  $gi$  and  $gf$  perpendicular to  $as$  are the intersections of this plane with the faces  $asb$ ,  $asc$ ; and  $gi$  and  $gf$  arc, at the same time, the true lengths of the two sides of the triangle in which the cutting plane meets the faces. Hence, if we consider  $gi$  as the base of this triangle, the vertex will be found, after the revolution at some point of the semi-circumference  $fpq$ , described from  $g$  as a centre, and with a radius  $gf$ .

With regard to the third side, its direction is made known by the intersection of the perpendicular plane to  $as$  with the third face: and since the inclination of this third face is known, it is easy to find the direction of this intersection. In the first place, we know it must pass through the point  $i$ ; to find a second point, draw  $go$  perpendicular to  $bs$ , and conceive a plane to be drawn through  $go$  perpendicular to the face  $asb$ . This plane will intersect the third face of the triedral angle in a right line which will make an angle with  $og$  equal to the given inclination. We may then make the revolution of this angle, in  $gom$ ; and if we draw  $gm$  perpendicular to  $go$ ,  $gm$  will be the true length of the perpendicular erected at  $g$  to the plane  $asb$ , and terminated in the plane of the third face. But this perpendicular lies entirely in the plane of the triangle which has  $gi$  for its base, and whose revolution on the plane  $asb$  we desire to find;

hence, when this triangle revolves around  $gi$ , its vertex must fall in the line  $ga$ , and be found at a distance from  $gi$ , such that  $gn = gm$ : therefore, if we draw a right line through the point  $i$  and  $n$ , we shall have the direction of the third side of the triangle.

When the line  $in$  meets the semi-circumference  $fpq$  in two points  $p$  and  $q$ , as in the figure, the problem admits of two solutions. For, if the plane of the circle  $fpq$  resume its original position, and we join the points  $p$  and  $q$  with the vertex  $s$ , it is evident that we shall have two diedral angles constructed from the conditions of the problem. Let us examine that which is determined by the point  $p$ . The line  $pi$  will be the distance of the point  $i$  from a point of the third edge: but the distance of the vertex  $s$  from the point of the third edge is equal to  $sf$ : hence, when we revolve the third edge around  $bs$ , this point will be found at the point of intersection  $p'$  of the arcs described with the radii  $ip$  and  $sf$ ; and, consequently, drawing  $sp'$ , we shall have  $bsp'$  for the third face. If we consider the triedral angle determined by the point  $q$ , and make like constructions, we shall have  $bsq'$  for the third face.

120. **Remarks.** If we let fall perpendiculars from  $p$  and  $q$  on  $gi$ , and then, through the foot of each perpendicular, draw perpendiculars to  $sb$ , these last lines must pass, respectively, through the points  $p'$  and  $q'$ .

The problem has only two solutions in the case in which the points  $p$  and  $q$  are both on the side  $in$ . When the point  $q$  falls on the side  $in'$ , there will be no solution for the point  $q$ . When the semicircle  $fpq$  is tangent to  $in$ , the two solutions are reduced to one; and finally, when the circle and line do not intersect, the problem becomes impossible.

### PROBLEM XXIX.

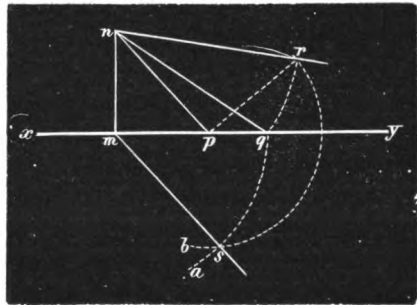
121. *To reduce the angle included between two right lines in space to the horizon.*

When we wish to make a map of a country, the prominent points of any part of the country are supposed to be connected by right lines, thus forming a system of rectilinear triangles; then, if all the points are in the same horizontal plane, the triangles may be readily traced on a map, upon a reduced scale; and

because of the similarity of triangles, an accurate representation will be presented by the map of the various points considered. But if the points considered are in different horizontal planes, as is the case in all mountainous countries, we suppose the triangles formed, by connecting the assumed points, to be projected on an assumed horizontal plane; and the triangles which result from this projection, are those which are transferred to the map by means of similar triangles.

It thus appears, that the map will not present the angles themselves which exist in space, but the projections of these angles on the assumed horizontal plane; so that, when it is required to *reduce the angle included between two right lines in space to the horizon*, the object is to find the horizontal projection of this angle. For this purpose, we measure on the surface of the ground, not only the angle which the given lines make with each other, but also the angle which each makes with the vertical passing through the angular point; and with these angles known, the problem may be solved.

Let  $L$  and  $L'$  represent the two given lines,  $A$  the angle which they make with each other, and  $V$  and  $V'$  the angles which they make, respectively, with the vertical. Draw, in the vertical plane of projection  $mn$  perpendicular to the ground line  $xy$ , and make the angle  $mnp$  equal to  $V$ ; and assume the vertical plane of projection as the plane of the angle  $V$ , which we are at liberty to do.



The line  $np$  is the line designated by  $L$ , and the other line  $L'$  must be represented, as passing through the point  $n$ , where it makes with  $mn$  the angle  $V'$ , and with  $np$  the angle  $A$ . It is required to find the horizontal projection of the angle  $A$ .

The line  $L$  meets the horizontal plane of projection at the point  $p$ ; and we have now to find the point in which the line  $L'$  intersects the same plane. For this purpose, draw the line  $nq$  making with  $mn$  the angle  $mnq = V'$ , and intersecting  $xy$  in  $q$ ;

then, from  $m$  as a centre, with a radius equal to  $mq$ , draw the indefinite arc  $qa$ . The line  $L$  must meet the horizontal plane at some point of this arc; so that it is only necessary to find the distance of this point of intersection from the point  $p$ . But this distance, considered in the plane of the angle  $A$ , is the base of a triangle whose sides are equal, respectively, to  $np$  and  $nq$ ; hence, if we make the angle  $pnr = A$ , and take  $nr = nq$ , the line  $pr$  will be equal to the distance sought. Then, from  $p$  as a centre, with a radius equal to  $pr$ , describe the arc  $rb$ , the point  $s$ , in which it intersects the arc  $qa$ , is the point in which the line  $L'$  meets the horizontal plane:  $mp$  and  $ms$  are therefore the projections of  $L$  and  $L'$ , and the angle  $pms$  is the required angle  $A$ .

#### PROBLEMS FOR THE EXERCISE OF THE STUDENT.

**Problem I.** Construct a plane which shall pass through the intersection of two given planes, and divide the angle of these planes equally.

**Prob. II.** Through a given point, draw a right line that shall make given angles with the planes of projection.

**Prob. III.** Through a given point, pass a plane that shall make given angles with the planes of projection.

**Prob. IV.** Knowing the traces of a plane, and the horizontal projection of the diagonal of a square lying in this plane, find the projections of the square.

**Prob. V.** Knowing the projections of the three edges of a triedral angle, find the traces of a plane that shall cut these three edges at given distances from the vertex.

**Prob. VI.** Circumscribe a sphere by a triangular pyramid.

**Prob. VII.** Inscribe a sphere in a triangular pyramid.



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